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DYNAMIC PERFORMANCE EVALUATION OF GENERAL THREE-STATE k-OUT-OF-n:G SYSTEMS WITH A NHCTM DEGRADATION PROCESS

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Abstract: This paper deals with the dynamic reliability analysis of three-state k-out-of-n:G systems in case of a non-homogeneous continuous time Markov (NHCTM) degradation process assumption. This assumption provides time-dependent transition rates between states of the components which is more realistic for real life applications. It is assumed that the components and the system have three states:perfect functioning, partial performance and complete failure. We study some performance characteristics of two type of three-state k-out-of-n systems by using the marginal and joint survival functions for the lifetime of those three-state k-out-of-n:G systems that consist of independent and nonidentical components. We consider permanent based representations for the survival probabilities. Numerical results for the performance characteristics of the two different types of systems are provided. The results are also supported with some graphical illustrations.

Key words: Multi-state system; Dynamic reliability analysis; Order statistics; Permanents; Non-homogeneous continuous time Markov process.

1. Introduction

Binary systems have only two working performance, success and failure. In dynamic reliability analysis of binary-state systems, a system is defined as "working" at time t, denoted by $\phi(t) = 1$, or failed at time t, denoted by $\phi(t) = 0$. Owing to the inefficiency to use binary-state systems in real life applications, multi-state systems are proposed instead and they are more practical to use in real life situations. They have more than just two levels of working efficiency. Thus a multi-state system and its components can have M $(M > 1)$ working states, from perfect functioning state $\phi(0) = M$ to less efficient states $\phi(t) \in \{M-1, M-2, \ldots, 1, 0\}$. Also different combinations of component states result in different working states of the system. For a detailed theory of multi-state modelling we refer to Kuo and Zuo [11] and Lisnianski and Levitin [12]. In the dynamic reliability analysis of multi-state systems a system or the components degradate into any lower state over time.One of the reliability measures which is commonly used in dynamic reliability analysis of multi-state systems is the probability that the system is in some intermediate state j or above at time t. This definition can be expressed as $R_j(t) = P\{\phi(t) \geq j\} = P\{T^{\geq j} > t\}, \forall j \epsilon [1, ..., M]$. Generally the degradation process of the multi-state systems or the components is assumed to have a homogeneous Markov process assumption(Ross [16]; Kolowrocki and Kwiatuszewska-Sarnecka [10]; Aven and Jensen [1]; Eryilmaz [3]; Liu and Kapur [13]; Yang and Xue [21]; Xue and Yang [20])or a non-homogeneous Markov process assumption(Liu and Kapur [14]; Shu et al. [18]). In this study we assume a non-homogeneous Markov process for the degradation process of the components According to this process, the length of time a machine tool stays in a certain state depends not only on the current state, but also on how long this process has been in the current state. This assumption is more realistic to reflect the performance degradation of multi-state systems. Recently, Iscioglu [9] dealt

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with the dynamic reliability evaluation of multi-state k-out-of-n:G systems with independent and identically distributed components under NHCTMP assumption.

In the present paper, we consider a system with n independent and nonidentical components. We assume that the system or the components can be in three-states, perfect functioning ("2"), partial working ("1") and complete failure ("0") . Three-state systems have been the topic of various reliability papers because of its simplicity (Guilani et al. [6], Hatoyama [7]). All components start to function at time $t = 0$, and then degradate into lower states with minor degradation. T_{1i} denotes the time that the ith component enters into state "0" and T_{2i} denotes the time that the ith component enters into state "1". It can be noted clearly that $T_{1i} \geq_{st} T_{2i}$. The random vectors (T_{1i}, T_{2i}) , $i = 1, 2, ..., n$ are independent with marginal distribution functions as;

$F_i(t; 1) = P\{T_{1i} \leq t\}$,

$F_i(t; 2) = P\{T_{2i} \leq t\}, \ \ i = 1, 2, ..., n.$

In this paper we study the multi-state k-out-of-n:G systems introduced by Huang et. al [8] and Tian et al. [19], where the systems and their components have three states each. The definitions of these systems are given in Section 2 of the paper. The survival functions represented in terms of permanents are also given in Section 2. The dynamic reliability measures for three-state k-out-of-n:G systems are achieved based on the assumption that the degradation of systems and the components follow NHCTM process. Therefore in Section 3 a general Markov process model is introduced in case of a multi-state element's degradation first. Then a detailed lifetime analysis is performed for three state systems in Section 4. Finally Section 5 concludes the paper with a brief summary and some possible further attempts.

2. Definitions for three-state k-out-of-n:G systems and survival functions

2.1. Definitions

Let $X_1, X_2, ..., X_n$ be the lifetimes of components in a binary k-out-of-n:G system, then $X_{n-k+1:n}$, $k = 1, 2, ..., n$ represents the lifetime of a k-out-of-n:G system where $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ are the ordered lifetimes of the components. The use of order statistics is not new in the reliability evaluation studies of binary-state and multi-state system structures. The lifetime properties of multi-state k-out-of-n systems based on order statistics have been first examined in Eryilmaz [4] and Eryilmaz and Xie [5] . He considered the definitions of the two commonly used generalized multi-state system structures (Huang et al. [8] and Tian et al. [19]) and represented their lifetimes in terms of order statistics. We also consider two definitions of generalized k-out-of-n multi-state structures where the systems and the components have three states, each. The definitions of those k-out-of-n:G structures are adapted to three states and presented below.

Definition 1. (Huang et al. [8]) A three-state k-out-of-n:G system I is in state "1" or above if at least k_1 components are in state "1" or above, or k_2 components are in state "2". It is in state "2" if at least k_2 components are in state "2".

Definition 2. (Tian et al. [19]) A three-state k-out-of-n:G system II is in state "1" or above if at least k_1 components are in state "1" or above. It is in state "2" if at least k_1 components are in state "1" or above, and at least k_2 components are in state "2".

In [4], lifetime properties of two multi-state k-out-of-n system structures have been considered when the components are independent and identical. In [5], dynamic reliability analysis of three-state k-out-of-n:G system structures, when the components are independent and nonidentical, is considered in case of permanent representations. Not only marginal survival functions but also joint survival functions are studied in [5]. In that study the lifetime of system I and system II are represented by letting $T_{r,1:n} \leq T_{r,2:n} \leq ... \leq T_{r,n:n}$ denote the order statistics corresponding to the rth $(r=1,2)$ coordinate of $(T_{11},T_{21}),(T_{12},T_{22}),...,(T_{1n},T_{2n})$ as follows;

The lifetime of System I ;

$$T_{I,1} = \max(T_{1,n-k_1+1:n}, T_{2,n-k_2+1:n}) \tag{2.1}$$

The lifetime of System I in state "2";

$$T_{I,2} = T_{2,n-k_2+1:n} \tag{2.2}$$

Similarly, the lifetime of System II;

$$T_{II,1} = T_{1,n-k_1+1:n} \tag{2.3}$$

and the lifetime of System II in state "2" ;

$$T_{II,2} = \min(T_{1,n-k_1+1:n}, T_{2,n-k_2+1:n}). \tag{2.4}$$

2.2. Survival functions

If $X_1, X_2, ..., X_n$ are order statistics of a set of independent random variables with continuous distribution functions $F_1, F_2, ..., F_n$, then the survival function of X_{n-r+1}can be represented in terms of permanents as;

$$P\{X_{n-r+1:n} > t\} = \sum_{i=r}^{n} \frac{1}{i!(n-i)!} Per \left[\underbrace{\bar{F}(t)}_{i}, \underbrace{F(t)}_{n-i} \right] \tag{2.5}$$

where $\bar{F}(t) = (\bar{F_1}(t), ..., \bar{F}_n(t))', F(t) = (F_1(t), ..., F_n(t))'$[2].

In order to find out the performance characteristics of two types of three-state k-out-of-n:G system structures given in Definitions (1) and (2) in case of NHCTM degradation process, let us first introduce some theorems given in [5] regarding some survival probabilities for the systems.

Theorem 1. *(Eryilmaz and Xie [5]) Let $(T_{1i}, T_{2i}), i = 1, ..., n$ be independent and nonidentical lifetime vectors of n components with $F_i(t;1) = P\{T_{1i} \leq t\}, F_i(t;2) = P\{T_{2i} \leq t\}$,and $H_i(t;2,1) = P\{T_{1i} > t, T_{2i} \leq t\} = F_i(t;2) - F_i(t;1), i = 1,2,...,n.$ Then for $t \geq 0$,*

$$P\{T_{1,n-k_1+1:n} > t, T_{2,n-k_2+1:n} > t\} = \sum_{i_1=k_1}^{n} \sum_{i_2=k_2}^{i_1} \frac{1}{i_2!(i_1-i_2)!(n-i_1)!} Per \left[\underbrace{\bar{F}(t;2)}_{i_2} \underbrace{H(t;2,1)}_{i_1-i_2} \underbrace{F(t;1)}_{n-i_1} \right] \tag{2.6}$$

where
$\overline{F}(t;2) = (\overline{F}_1(t;2), ..., \overline{F}_n(t;2))'$
$H(t;2,1) = (F_1(t;2) - F_1(t;1), ..., F_n(t;2) - F_n(t;1))'$
$F(t;1) = (F_1(t;1), ..., F_n(t;1))'$

Using equation (2.5) and Theorem 1, one can obtain the survival functions of lifetime random variables (2.1)-(2.4) as;

$$P\{T_{I,j} > t\} = \begin{cases} p^1_{k_1,n}(t) + p^2_{k_2,n}(t) - p^{1,2}_{k_1,k_2,n}(t) & \text{if } j=1 \\ p^2_{k_2,n}(t) & \text{if } j=2, \end{cases} \tag{2.7}$$

and

$$P\{T_{II,j} > t\} = \begin{cases} p^1_{k_1,n}(t) & \text{if } j = 1 \\ p^{1,2}_{k_1,k_2,n}(t) & \text{if } j = 2, \end{cases} \tag{2.8}$$

where
$p^1_{k_1,n}(t) = \sum_{i=k_1}^n \frac{1}{i!(n-i)!} Per \left[\underbrace{\bar{F}(t;1)}_{i} \underbrace{F(t;1)}_{n-i} \right]$,
$p^2_{k_2,n}(t) = \sum_{i=k_2}^n \frac{1}{i!(n-i)!} Per \left[\underbrace{\bar{F}(t;2)}_{i} \underbrace{F(t;2)}_{n-i} \right]$,
and
$p^{1,2}_{k_1,k_2,n}(t) = P\{T_{1,n-k_1+1:n} > t, T_{2,n-k_2+1:n} > t\}$.

The system state probabilities of three-state k-out-of-n:G systems I and II can easily be represented by using equations (2.7)-(2.8) respectively as;

$$P\{\phi_I(t) = j\} = \begin{cases} 1 - p^1_{k_1,n}(t) - p^2_{k_2,n}(t) + p^{1,2}_{k_1,k_2,n}(t) & \text{if } j = 0 \\ p^1_{k_1,n}(t) - p^{1,2}_{k_1,k_2,n}(t) & \text{if } j = 1 \\ p^2_{k_2,n}(t) & \text{if } j = 2 \end{cases} \tag{2.9}$$

$$P\{\phi_{II}(t) = j\} = \begin{cases} 1 - p^1_{k_1,n}(t) & \text{if } j = 0 \\ p^1_{k_1,n}(t) - p^{1,2}_{k_1,k_2,n}(t) & \text{if } j = 1 \\ p^{1,2}_{k_1,k_2,n}(t) & \text{if } j = 2 \end{cases} \tag{2.10}$$

where $\phi_i(t)$ denotes the state of the three state k-out-of-n:G system i, at time $t, i = I, II$.

THEOREM 2. *(Eryilmaz and Xie [5]) Let $(T_{1i}, T_{2i}), i = 1, ..., n$ be independent and nonidentical lifetime vectors of n components with $F_i(t;1) = P\{T_{1i} \le t\}, F_i(t;2) = P\{T_{2i} \le t\}$, and*
$H_i(t;2,1) = P\{T_{1i} > t, T_{2i} \le t\} = F_i(t;2) - F_i(t;1), i = 1, 2, ..., n$. Then
$H_i(t;2,1) = P\{T_{1i} > t, T_{2i} \le t\} = F_i(t;2) - F_i(t;1), i = 1, 2, ..., n$. Then

$$P\{T_{1,n-k_1+1:n} < T_{2,n-k_2+1:n}\}$$

$$= \sum_{i=1}^{n} \sum_{j=n-k_1+1}^{n-k_2} \frac{1}{(k_2-1)!(n-k_2-j)!j!}$$

$$\times \int_0^\infty Per \left[\underbrace{\bar{F}_{-i}(t;2)}_{k_2-1} \underbrace{H_{-i}(t;2,1)}_{n-k_2-j} \underbrace{F_{-i}(t;1)}_{j} \right] dF_i(t;2)$$

where
$\bar{F}_{-i}(t;2) = (\overline{F}_1(t;2), ..., \overline{F}_{i-1}(t;2), \overline{F}_{i+1}(t;2), ..., \overline{F}_n(t;2))'$
$H_{-i}(t;2,1) = (H_1(t;2,1), ..., H_{i-1}(t;2,1), H_{i+1}(t;2,1), ..., H_n(t;2,1))'$
$F_{-i}(t;1) = (F_1(t;1), ..., F_{i-1}(t;1), F_{i+1}(t;1), ..., F_n(t;1))'$.

The proofs of Theorem 1 and 2 can be found in Eryilmaz and Xie [5].

3. General Markov Process Model and Multi-state tool wear with minor degradation

Let $\phi(t) \in \{0,1,2,\ldots,M\}$ be a discrete-state continuous-time stochastic process and denote the state of a multi-state element at time t and take values of $j_1, j_2, ...j_n$. This process is called a Markov chain if its conditional probability mass function satisfies,

$$\begin{aligned} P\{\phi(t_n) = j_n \mid \phi(t_{n-1}) = j_{n-1}, ..., \phi(t_2) = j_2, \phi(t_1) = j_1\} \\ = P\{\phi(t_n) = j_n \mid \phi(t_{n-1}) = j_{n-1}\} \end{aligned} \tag{3.1}$$

for $t_1 \le t_2 \le ... \le t_{n-1} \le t_n$ where $j_i \in \{0,1,\ldots,M\}, i = 1,2,...,n$ and $j_1 \le j_2 \le ... \le j_{n-1} \le j_n$. Then if $t_{n-1} = t$ and $t_n = t + \triangle t$, the expression (3.1) is simplified to

$$P\{\phi(t+\triangle t) = j \mid \phi(t) = i\} = P_{i,j}(t,\triangle t)$$

for $i,\ j \in \{0,1,\ldots,M\}$. These conditional probabilities are called the transition probabilities of NHCTM process (Ross [16]; Lisnianski and Levitin [12]). The transition probabilities satisfy

$$P_{i,j}(t,\triangle t) \ge 0 \ \text{ and } \sum_{j=0}^{M} P_{i,j}(t,\Delta t) = 1 \ \ (i,j = 0,1,...,M)$$

for ,$\Delta t > 0$. The Chapman-Kolmogorov equation, which follows the Markov property and the rule for total probability, is given by (Ross [16])

$$P_{i,j}(t+\Delta t) = \sum_{k=0}^{M} P_{i,k}(t) P_{k,j}(t,\Delta t) \tag{3.2}$$

for $t, \Delta t > 0$. From (3.2), we have

$$P_{i,j}\left(t+\Delta t\right) - P_{i,j}(t) = -P_{i,j}(t) \sum_{k=0,k\neq j}^{M} P_{i,k}(t,\Delta t) + \sum_{k=0,k\neq j}^{M} P_{i,k}(t) P_{k,j}(t,\Delta t) \tag{3.3}$$

Also let

$$\begin{aligned} \lambda_{i,j}(t) &= \lim_{\Delta t \to 0} \frac{P\{\phi(t+\Delta t) = j \mid \phi(t) = i\}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{P_{i,j}(t,\Delta t)}{\Delta t} \end{aligned}$$

and

$$\lambda_{j,j}(t) = -\sum_{k=0,k\neq j}^{M} \lambda_{j,k}(t)$$

where $\lambda_{i,j}(t)$ is the transition degradation rate from state i to state j and $\lambda_{j,j}(t)$ is the transition degradation rate from state j to state j. Then for the deterioration processes of machine elements followed by a NHCTM process, the state equations of elements are obtained by dividing (3.3) by Δt and letting $\Delta t \to 0$, and expressed as,

$$P^{'}_{i,j}(t) = -P_{i,j}(t) \sum_{k=0,k\neq j}^{M} \lambda_{j,k}(t) + \sum_{k=0,k\neq j}^{M} P_{i,k}(t)\lambda_{k,j}(t) \tag{3.4}$$

Let us now assume that $\phi(0) = i = M$, that the machine element is in state M at time $t = 0$. Then, the notation of the state equations shown in (3.4) can be simplified by omitting the index i, that is

$$P^{'}_{j}(t) = -P_{j}(t) \sum_{k=0,k\neq j}^{M} \lambda_{j,k}(t) + \sum_{k=0,k\neq j}^{M} P_{k}(t)\lambda_{k,j}(t) \tag{3.5}$$

where $P_M(0) = 1$ and $P_k(0) = 0$ for $k \neq m$ and $P_j(t)$ is the probability that the machine element is in state j at time t.

Let $\phi(t) \in \{0, 1, 2\}$ be a discrete-state continuous-time stochastic process which represents the state of a three-state element at time t. In this study NHCTM process is used to describe the age-dependent performance degradation process for both the three-state elements and the system. A three-state deterioration process followed by a NHCTM process is defined as a major degradation process if the process is capable of degrading directly to any lower state from state i to state j for all $j \leq i-1$ and $i \in \{2, 1\}$ during a transition. However if the process is capable of degrading to only the nearest state from its current state during a transition, then it is defined as a minor degradation process. Thus in this paper both processes are considered for both the elements and the system. In order to find the state probabilities of the elements we need to solve equation (3.5). It is more practical to express the equation (3.5) as a matrix notation as,

$$[P^{'}(t)]^T = \Lambda(t)[P(t)]^T \tag{3.6}$$

where $P(t) = [P_2(t), P_1(t), P_0(t)]$, $P^{'}(t) = [P^{'}_2(t), P^{'}_1(t), P^{'}_0(t)]$ and the transition matrix,

$$\Lambda(t) = \begin{bmatrix} \lambda_{2,2}(t) & 0 & 0 \\ \lambda_{2,1}(t) & \lambda_{1,1}(t) & 0 \\ \lambda_{2,0}(t) & \lambda_{1,0}(t) & 0 \end{bmatrix} \tag{3.7}$$

Note that state 0 is the absorbing state. Thus the corresponding column equals to zero. By solving the equation (3.6), the state probabilities are easily obtained. Also the probability of the system to be at state 0 at time t, can be obtained from $P_0(t) = 1 - P_1(t) - P_2(t)$. Matrix notation can be used practically to find the state distributions of components. Many researchers have also proposed some methods to get the solution such as; Liu and Kapur [14], [15] used enumerative method to get the solution, Shu et al. [18] presented a state-by-integration method based on Liu and Kapur [14] with some modifications and extensions. Also recently Sheu and Zhang [17] proposed a more efficient recursive approach to find the state distribution of components or the system. This recent approach also took the advantage of the structure of the infinitesimal matrix. According to this approach for a three-state system, the probability of the element or the system to be at state i ($i \in \{2, 1, 0\}$) at time t is obtained by;

$$P_i(t) = P\{\phi(t) = i\} = \sum_{k=i+1}^{2} \int_0^t P_k(\tau_{3-k})\lambda_{k,i}(\tau_{3-k}) \tag{3.8}$$
$$\times \exp[-\int_{\lambda_{3-k}}^{t} \sum_{j=0}^{i-1} \lambda_{i,j}(s)ds]d\tau_{3-k}$$

where $i = 1, 0$. In equation (3.8), the transition occurs at some unknown time τ_{3-k} and the system stays in state i for the remaining $t - \tau_{3-k}$. Also the conditions, $P_2(0) = P\{\phi(0) = 2\} = 1$ and $P_k(0) = P\{\phi(0) = k\} = 0$ for $k = 1, 0$, are assumed.

4. Performance evaluation of a three-state system with a numerical illustration

Marginal and joint survival probabilities for the lifetime of two different three state k-out-of-n:G systems that consist of independent and nonidentical components are evaluated in this section. Throughout the paper, systems are assumed to have independent and nonidentical components, and each component and the system have three possible states that are given by $0, 1, 2$ where '0' is the complete failure state and '2' is the perfect functioning state. Since the components are independent and nonidentical , the random vectors (T_{1i}, T_{2i}), $i = 1, ..., n$ are independent with marginals $F_i(t;1) = P\{T_{1i} \le t\}$ and $F_i(t;2) = P\{T_{2i} \le t\}$. Also we assume that the degradation in the components follows a NHCTM process. If the instantaneous degradation rates from state "2" to "1" and "1" to "0" are denoted respectively by $\lambda_{2,1}^{(i)}$ and $\lambda_{1,0}^{(i)}$, $i = 1, 2, ..., n$, then

$$F_i(t;2) = P\{T_{2i} \le t\} = 1 - e^{-\int_0^t (\lambda_{2,1}^{(i)}(s)+\lambda_{2,0}^{(i)}(s))ds} \tag{4.1}$$

$$F_i(t;1) = P\{T_{1i} \le t\} = 1 - [\int_0^t e^{-\int_0^{\tau_1} (\lambda_{2,1}^{(i)}(s)+\lambda_{2,0}^{(i)}(s))ds} \lambda_{2,1}^{(i)}(\tau_1) e^{-\int_{\tau_1}^t \lambda_{1,0}^{(i)}(s)ds} d\tau_1 + e^{-\int_0^t (\lambda_{2,1}^{(i)}(s)+\lambda_{2,0}^{(i)}(s))ds}] \tag{4.2}$$

$\lambda_{2,1}^{(i)} \ne \lambda_{1,0}^{(i)}, i = 1, 2, ..., n$. Equations (4.1) and (4.2) can be calculated using the method in Liu and Kapur [14] or Shue and Zhang [17]. These equations are used when major degradation process is discussed. In case of minor degradation, equations (4.1) and (4.2) are simplified by omitting $\lambda_{2,0}^{(i)}(s)$ from the equations because the elements in the system are only able to degradate to the nearest states from their current states.

Consider a system with $n = 3$ components with $k_1 = 2$ and $k_2 = 1$. Let us first consider three cases for the numerical illustration. For Case I, II and III we let the lifetime distribution of each component be non-identically distributed. Thus the transient degradation rates for the lifetimes of components following $Weibull(1/(j_1 - \theta_i j_2), \beta)$ for $j_1 = 2$, $j_2 \in \{2, 1, 0\}$ and $i = 1, 2, ..., n$ are given by;

$$\lambda_{j_1,j_2}(t) = \frac{\beta t^{\beta-1}}{(j_1 - \theta_i j_2)^\beta}$$

where $j_1 \in \{1, 2\}$, $0 \le j_2 \le j_1 - 1$, and

$$\lambda_{j,j}(t) = -\sum_{k=0,k\ne j}^{j} \frac{\beta t^{\beta-1}}{(j - \theta_i k)^\beta}, \quad j \in \{2, 1, 0\}, i = 1, 2, ..., n.$$

Then for different values of θ_i and β, the instantaneous degradation rates of components are given in Table 1.

TABLE 1. Instantaneous degradation rates of components

	θ_i	β	$\lambda_{2,2}^{(i)}(t)$	$\lambda_{2,1}^{(i)}(t)$	$\lambda_{2,0}^{(i)}(t)$	$\lambda_{1,1}^{(i)}(t)$	$\lambda_{1,0}^{(i)}(t)$
Case I	0.5	3	$-1.2639t^2$	$0.8889t^2$	$0.375t^2$	$-3t^2$	$3t^2$
	0.4	3	$-1.1074t^2$	$0.7324t^2$	$0.375t^2$	$-3t^2$	$3t^2$
	0.3	3	$-0.9856t^2$	$0.6106t^2$	$0.375t^2$	$-3t^2$	$3t^2$
Case II	0.6	3	$-1.4683t^2$	$1.0933t^2$	$0.375t^2$	$-3t^2$	$3t^2$
	0.5	3	$-1.2639t^2$	$0.8889t^2$	$0.375t^2$	$-3t^2$	$3t^2$
	0.4	3	$-1.1074t^2$	$0.7324t^2$	$0.375t^2$	$-3t^2$	$3t^2$
Case III	0.5	4	$-1.0412t^3$	$0.7901t^3$	$0.25t^3$	$-4t^3$	$4t^3$
	0.4	4	$-0.8604t^3$	$0.6104t^3$	$0.25t^3$	$-4t^3$	$4t^3$
	0.3	4	$-0.7289t^3$	$0.4789t^3$	$0.25t^3$	$-4t^3$	$4t^3$

The survival functions of lifetime random variables (2.1)-(2.4) for the three-state k-out-of-n:G system I are;

$P\{T_{I,1} > t\} = p_{2,3}^1(t) + p_{1,3}^2(t) - p_{2,1,3}^{1,2}(t)$

$P\{T_{I,2} > t\} = p_{1,3}^2(t)$,

and for the three-state k-out-of-n:G system II;

$P\{T_{II,1} > t\} = p_{2,3}^1(t)$,

$P\{T_{II,2} > t\} = p_{2,1,3}^{1,2}(t)$.

where

$$p_{2,3}^1(t) = P\{T_{1,2:3} > t\} = \overline{F}_1(t;1)\overline{F}_2(t;1)F_3(t;1) + \overline{F}_1(t;1)\overline{F}_3(t;1)F_2(t;1) + \overline{F}_2(t;1)\overline{F}_3(t;1)F_1(t;1) + \overline{F}_1(t;1)\overline{F}_2(t;1)\overline{F}_3(t;1)$$

$$p_{1,3}^2(t) = P\{T_{2,3:3} > t\} = \overline{F}_1(t;2)F_2(t;2)F_3(t;2) + \overline{F}_2(t;2)F_1(t;2)F_3(t;2) + \overline{F}_3(t;2)F_1(t;1)F_2(t;2) + \overline{F}_1(t;2)\overline{F}_2(t;2)F_3(t;2) + \overline{F}_1(t;2)\overline{F}_3(t;2)F_2(t;2) + \overline{F}_2(t;2)\overline{F}_3(t;2)F_1(t;2) + \overline{F}_1(t;2)\overline{F}_2(t;2)\overline{F}_3(t;2)$$

and

$$p_{2,1,3}^{1,2}(t) = P\{T_{1,2:3} > t, T_{2,3:3} > t\} = \overline{F}_1(t;2)\overline{F}_2(t;1) + \overline{F}_3(t;2)\overline{F}_2(t;1) - \overline{F}_3(t;2)\overline{F}_1(t;2) + \overline{F}_2(t;2)\overline{F}_1(t;1) - \overline{F}_2(t;2)\overline{F}_3(t;2) - \overline{F}_2(t;2)\overline{F}_1(t;2)F_3(t;2) + \overline{F}_1(t;2)\overline{F}_3(t;1)F_2(t;1) + \overline{F}_3(t;2)\overline{F}_1(t;1)F_2(t;1) + \overline{F}_2(t;2)\overline{F}_3(t;1)F_1(t;1).$$

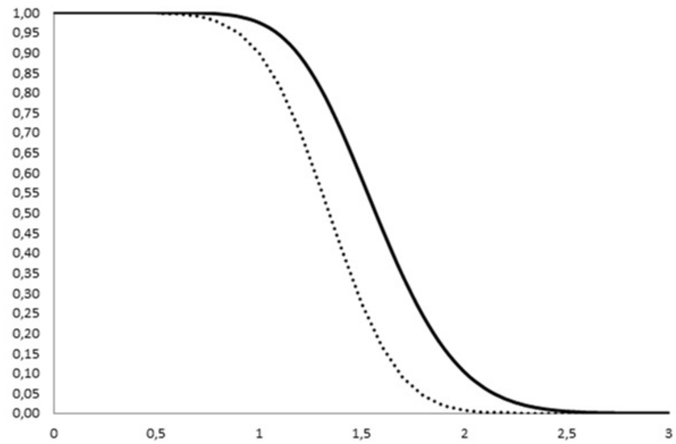
Using Eqs. (4.1) and (4.2), we can compute survival functions and means of $T_{I,1}, T_{I,2}, T_{II,1}$ and $T_{II,2}$.

In Figs. 1 and 2, we plot the survival functions $P\{T_{I,1} > t\}, P\{T_{II,1} > t\}$ and $P\{T_{I,2} > t\}, P\{T_{II,2} > t\}$ respectively for Case II. In both figures, the solid line is for system I and the dashed line is for system II. As expected, $T_{I,i}$ is stochastically greater than $T_{II,i}$ for $i = 1,2$. Related survival probabilities for both the systems are lower in case of major degradation compared to minor degradation as well.

In Table 2, we compute the mean lifetimes of two types of three-state systems for the three cases with different instantaneous degradation rates given in Table 1. For all the Cases, expected lifetime of system II in each state is lower than the expected lifetime of system I in each state. Also for both systems, the expected lifetime in state 1 is greater than the expected lifetime in state 2. The expected lifetimes of both systems are lower in case of major degradation compared to minor degradation.

In Table 3, we compute $P\{T_{1,n-k_1+1:n} < T_{2,n-k_2+1:n}\}$ for all cases given in Table 1. From the Table we observe that an increase(decrease) in θ_i parameters leads to an increase(decrease) in $\lambda_{2,1}^{(i)}$s

(a) Major Degradation



(b) Minor Degradation

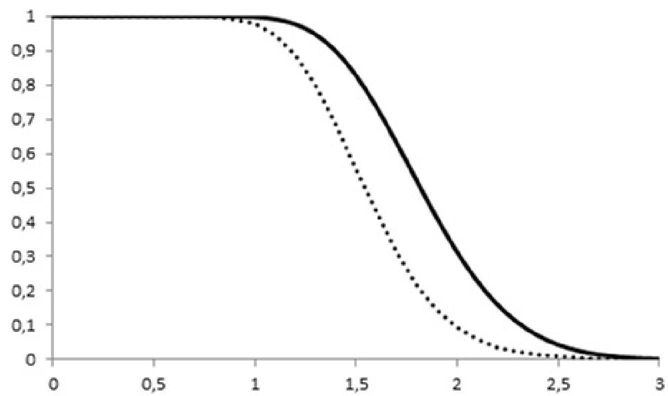


FIGURE 1. Plots of survival functions $P\{T_{I,1} > t\}$ and $P\{T_{II,1}\}$ when $n = 3$, $k_1 = 2$, $k_2 = 1$.

TABLE 2. Mean Lifetimes of system I and system II when $n = 3$, $k_1 = 2$, $k_2 = 1$

	Major Degradation			
	$E(T_{I,1})$	$E(T_{I,2})$	$E(T_{II,1})$	$E(T_{II,2})$
Case I	1.6501	1.6347	1.3773	1.3620
Case II	1.5854	1.5650	1.3412	1.3208
Case III	1.6549	1.6493	1.4127	1.4071
	Minor Degradation			
	$E(T_{I,1})$	$E(T_{I,2})$	$E(T_{II,1})$	$E(T_{II,2})$
Case I	1.9461	1.8830	1.6325	1.5694
Case II	1.8411	1.7663	1.5667	1.4919
Case III	1.8344	1.8047	1.5649	1.5352

and this leads to a decrease (increase) in the corresponding probability. We also observe that an increase(decrease) in β parameters when the θ_i parameters are same, leads to a decrease(increase) in $\lambda_{2,1}^{(i)}$s and an increase(decrease) in $\lambda_{1,0}^{(i)}$s . Thus this leads to an increase (decrease) in the cor-

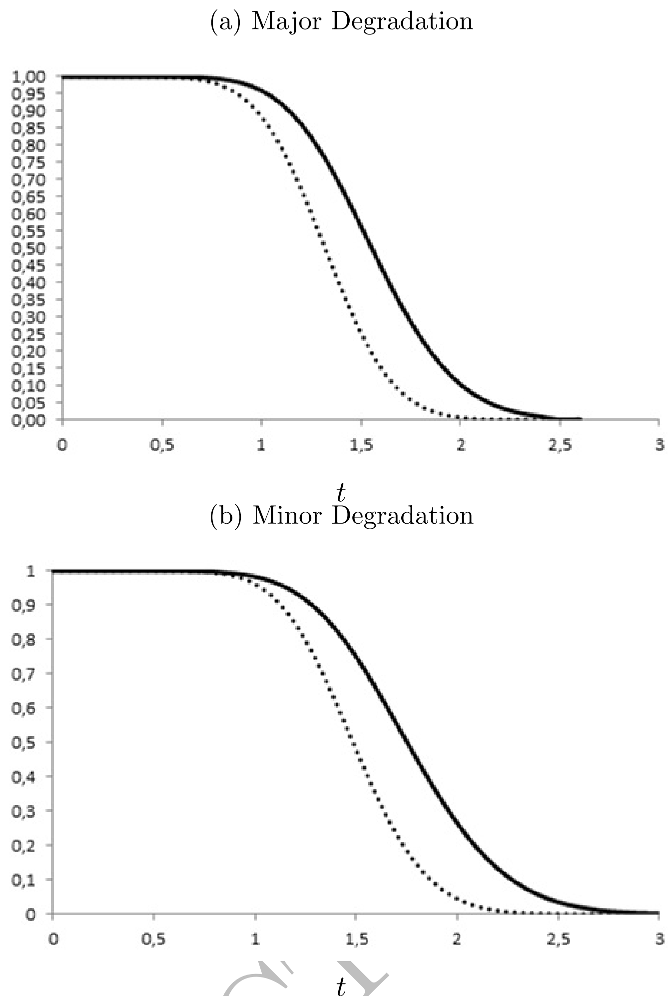


Figure 2. Plots of survival functions $P\{T_{I,2} > t\}$ and $P\{T_{II,2}\}$ when $n = 3$, $k_1 = 2$, $k_2 = 1$.

responding probability. Also the related probability is lower when major degradation is discussed compared to minor degradation in all the cases.

Table 3. $P\{T_{1,n-k_1+1:n} < T_{2,n-k_2+1:n}\}$ when $n = 3, k_1 = 2, k_2 = 1$

	Major Degradation	Minor Degradation
Case I	0.7756	0.7783
Case II	0.7369	0.7386
Case III	0.8528	0.8571

5. Conclusions

In this paper, we have studied some performance measures for two multi-state k-out-of-n:G systems consisting of independent and nonidentical components. The order statistics related results for lifetime vectors of those systems have been used in evaluating the related survival probabilities or the expected values in case of NHCTM degradation processes of the components. Thus we have dealt with time dependent degradation rates in the calculations which makes the problem more realistic for the applications in the dynamic reliability analysis. It has been known that the order statistics distribution theory is not new for the dynamic reliability analysis for binary k-out-of-n:G systems. However in multi-state k-out-of-n:G systems they have been recently used in some of the papers. Thus the concept of order statistics and especially permanents are first used for the dynamic reliability evaluation of systems with nonidentical components in case of NHCTM processes in this paper. For the future research problems, one possible topic can be modeling and analysis of three-state k-out-of-n:G systems with dependent components which also have NHCTM degradation processes.

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