

# BAYESIAN PREDICTION OF PROGRESSIVELY FIRST-FAILURE-CENSORED ORDER STATISTICS BASED ON $k$ -RECORD VALUES FROM WEIBULL DISTRIBUTION

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**Abstract:** Prediction on the basis of censored data has an important role in lifetime studies. This paper discusses Bayesian two-sample prediction of progressively first-failure-censored order statistics coming from a future sample based on observed  $k$ -record values from two-parameter Weibull distribution. Bayesian interval predictions are obtained based on a continuous-discrete joint prior for the unknown two parameters. Moreover, the Bayesian point predictors are investigated under symmetric and asymmetric loss functions. Finally, the estimated risks of the various point predictors obtained are compared using the Monte Carlo method.

**Key words:** Bayesian prediction;  $k$ -record; Progressive first-failure censoring; Weibull distribution

## 1. Introduction

Prediction of the future samples based on current sample information (called informative sample) is very interesting topic in Statistics. Here, the informative sample and the future sample are supposed stochastically independent. The aim of this paper is to discuss the predicting progressively first-failure-censored order statistics arising from a future sample based on observed  $k$ -records. First, we present a brief description to the progressively first-failure-censored samples and the  $k$ -records.

Censoring is usual in lifetime data due to time and cost restrictions. There are various types of censoring in survival analysis and progressive censoring is one of the most common for consideration. This censoring allows the experimenter to remove units from a life test at various stages during the experiment. For a comprehensive review of theory, methods and applications of the progressive censoring, we refer the reader to Balakrishnan and Aggarwala [7], Balakrishnan [6] and the references contained therein. Also, recently a book due to Balakrishnan and Cramer [8] offers a thorough and updated guide to the theory and methods of progressive censoring along with its practical applications to reliability and survival analyses. Progressive first-failure censoring, introduced by Wu and Kuş [30], is a type of progressive censoring in which  $n$  disjoint groups with  $s$  identical units within each group ( $N = n \times s$ ) are placed on a life test at time zero. Suppose the random variables  $X_1, \dots, X_N$  denote their corresponding lifetimes. The life test is terminated at the time of  $m$ -th failure. When the  $i$ -th unit fails ( $i = 1, 2, \dots, m - 1$ ), randomly selected  $R_i$  groups

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TABLE 1. The  $k$ -record times and  $k$ -record values for the insulating fluid data set when  $k = 2$ .

$r$	1	2	3	4	5	6	7	8
$T_{r^{(k)}}$	2	5	6	9	10	11	18	19
$U_{r^{(k)}}$	0.96	4.15	8.01	8.27	31.75	32.52	33.91	36.71

TABLE 2. The  $k$ -record times and  $k$ -record values for the insulating fluid data set when  $k = 3$ .

$r$	1	2	3	4	5	6	7	8	9	10
$T_{r^{(k)}}$	3	4	5	6	7	9	10	11	18	19
$U_{r^{(k)}}$	0.19	0.78	0.96	4.15	7.35	8.01	8.27	31.75	32.52	33.91

1 and the group in which the  $i$ -th failure is observed are removed from the test. When the  $m$ -th  
 2 failure occurs, all of the remaining groups are removed from the test. The  $m$  observed failure times  
 3 denoted by  $X_{1:m:n:s}^{\mathbf{R}} < \dots < X_{m:m:n:s}^{\mathbf{R}}$  are the progressively first-failure-censored order statistics with  
 4 pre-determined censoring scheme  $\mathbf{R} = (R_1, \dots, R_m)$ . Note that: (1) for  $s = 1$ , the progressive first-  
 5 failure-censoring scheme is reduced to the case of progressive Type-II censoring, (2) if  $R_i = 0$  for  
 6  $i = 1, 2, \dots, m$ , we have the first-failure censoring, (3) if  $s = 1$ ,  $R_i = 0$  for  $i = 1, 2, \dots, m - 1$  and  
 7 hence  $R_m = n - m$ , this scheme is reduced to the Type-II censoring and (4) if  $s = 1$  and  $R_i = 0$  for  
 8  $i = 1, 2, \dots, m$ , this scheme is simplified to the complete sample. According to Wu and Kuş [30],  
 9 “Although more units are used (only  $m$  of  $N$  units fail) in the progressive first-failure-censoring  
 10 plan than in others, it has advantages in terms of reducing test cost and test time”.

Let  $\{Y_i, i \geq 0\}$  be a sequence of continuous random variables. Then,  $Y_j$  will be called an upper  
 record value (or simply record value) if  $Y_j > Y_i$  for every  $i < j$ . A similar definition can be proposed  
 for lower record values. Analogously, an upper  $k$ -record value (or simply  $k$ -record value) is defined  
 in terms of the  $k$ -th largest  $Y$  yet seen. Precisely, let  $T_{1^{(k)}} = k$ ,  $U_{1^{(k)}} = Y_{1:k}$  and for  $r \geq 2$ ,

$$T_{r^{(k)}} = \min\{j : j > T_{r-1^{(k)}}, Y_j > Y_{T_{r-1^{(k)}}-k+1:T_{r-1^{(k)}}}\},$$

11 where  $Y_{i:n}$  denotes the  $i$ -th order statistic in a random sample of size  $n$ . In the literature,  $\{T_{r^{(k)}}, r \geq$   
 12  $1\}$  is said to be the  $k$ -record times sequence. Thus, the sequence of  $k$ -record values is defined by  
 13  $U_{r^{(k)}} = Y_{T_{r^{(k)}}-k+1:T_{r^{(k)}}}$  for  $r \geq 1$ . For the special case  $k = 1$ , the usual record values are obtained.  
 14 Dziubdziela and Kopocinski [13] showed that  $\{U_{r^{(k)}}, r \geq 1\}$  arising from a sequence of independent  
 15 and identically distributed (IID) random variables with the common distribution function (DF)  
 16  $F$  is distributed as the usual record values (i.e.  $k = 1$ ) coming from a sequence of IID random  
 17 variables with the common DF  $1 - (1 - F)^k$ . Various applications of  $k$ -record values can be found  
 18 in the literature; See, e. g. Arnold *et al.* [5], Kamps [16, 17] and Nevzorov [22]. For example, the  
 19  $r$ -th  $k$ -record value can be regarded as the life length of a  $k$ -out-of- $T_{r^{(k)}}$  system. As an illustration,  
 20 consider the following data set given by Lawless [19, p. 185]:

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91	32.52	3.16	4.85	2.78
4.67	1.31	12.06	36.71	72.89									

21 These data contains times to breakdown of an insulating fluid between electrodes recorded at  
 22 34 kilovolts. For  $k = 2$  and  $k = 3$ , the  $k$ -record times and  $k$ -record values are reported in Tables 1  
 23 and 2.

24 Statistical predictions via a Bayesian approach have been considered in the literature; See, for  
 25 example, Madi and Raqab [20], Ali Mousa and Jaheen [3], Ali Mousa and Al-Sagheer [4], Raqab  
 26 *et al.* [23], Kundu and Howlader [18], Ahmadi *et al.* [1, 2], Ghafoori *et al.* [14], Huang and Wu  
 27 [15], Balakrishnan and Shafay [9], Saeidi *et al.* [24] and Doostparast [11]. Also, Soliman *et al.*

[27] considered point and interval Bayesian predictions on the basis of general progressively Type-II censored data coming from Weibull model under symmetric and asymmetric loss functions. Recently, Shafay *et al.* [25] discussed the problem of predicting the future sequential order statistics based on observed multiply Type-II censored samples of sequential order statistics from one- and two-parameter exponential distributions.

In this paper, we suppose that the parent population follows the Weibull distribution with DF

$$F(x; \alpha, \beta) = 1 - \exp\{-\alpha x^\beta\}, \quad x > 0, \alpha > 0, \beta > 0. \quad (1.1)$$

Hence, the density function of the Weibull distribution is

$$f(x; \alpha, \beta) = \alpha \beta x^{\beta-1} \exp\{-\alpha x^\beta\}, \quad x > 0, \alpha > 0, \beta > 0. \quad (1.2)$$

The Weibull distribution is one of the most popular distributions that is extensively used for modelling the failure data. For more details on applications of the Weibull distribution, see, for example, Murthy *et al.* [21] and references therein.

The rest of this paper is organized as follows. In Sections 2 and 3, Bayesian interval and point predictors for progressively first-failure-censored order statistics arising from a future sample based on observed  $k$ -record values are derived. To do this, we consider two cases for the shape parameter  $\beta$  in the DF (1.1). The performance analysis of the obtained predictors is carried out by conducting a simulation study in Section 4. Finally, we use a real data set from Dunsmore [12] for illustrating the inferential methods developed here.

## 2. Bayesian prediction interval

Let  $U_{1(k)} < \dots < U_{r(k)}$  be the first  $r$   $k$ -record values from a continuous population with density function and DF  $f(\cdot|\boldsymbol{\theta})$  and  $F(\cdot|\boldsymbol{\theta})$ , respectively, where  $\boldsymbol{\theta}$  is a vector of parameters. Following Arnold *et al.* [5], the associated likelihood function (LF) of the observed data  $\mathbf{u} = (u_1, \dots, u_r)$  is reduced to

$$L(\boldsymbol{\theta}; \mathbf{u}) = k^r (1 - F(u_r|\boldsymbol{\theta}))^k \prod_{j=1}^r \frac{f(u_j|\boldsymbol{\theta})}{1 - F(u_j|\boldsymbol{\theta})}.$$

Under the Weibull distribution, the LF becomes

$$L(\alpha, \beta; \mathbf{u}) = (k\alpha\beta)^r \prod_{j=1}^r u_j^{\beta-1} \exp\{-k\alpha u_r^\beta\}, \quad 0 < u_1 < \dots < u_r < \infty. \quad (2.1)$$

On the other hand, let  $X_{1:m:n:s}^{\mathbf{R}}, \dots, X_{m:m:n:s}^{\mathbf{R}}$  be a progressively first-failure-censored sample from a continuous population with density function and DF  $f(\cdot|\boldsymbol{\theta})$  and  $F(\cdot|\boldsymbol{\theta})$ , respectively. It can be seen that  $X_{1:m:n:s}^{\mathbf{R}}, \dots, X_{m:m:n:s}^{\mathbf{R}}$  can be viewed as a progressively type II censored sample from a continuous population with density function and DF  $sf(\cdot|\boldsymbol{\theta})(1 - F(\cdot|\boldsymbol{\theta}))^{s-1}$  and  $1 - (1 - F(\cdot|\boldsymbol{\theta}))^s$ , respectively. Then, the marginal density function of  $X_{i:m:n:s}^{\mathbf{R}}$ , ( $1 \leq i \leq m$ ) is given by (see, for example, Balakrishnan and Aggarwala [7])

$$f_{X_{i:m:n:s}^{\mathbf{R}}}(x_i|\boldsymbol{\theta}) = c_{i-1} s f(x_i|\boldsymbol{\theta}) (1 - F(x_i|\boldsymbol{\theta}))^{s-1} \sum_{j=1}^i a_j (1 - F(x_i|\boldsymbol{\theta}))^{s(\gamma_j-1)}, \quad (2.2)$$

where

$$\gamma_j = \sum_{l=j}^m (R_l + 1) = n - \sum_{l=1}^{j-1} (R_l + 1), \quad c_{i-1} = \prod_{j=1}^i \gamma_j,$$

$$a_j = \prod_{l=1, l \neq j}^i \frac{1}{\gamma_l - \gamma_j}, \quad 1 < i \leq m \quad \text{and} \quad a_1 = 1.$$

1 Upon substituting Equations (1.2) and (1.1) into Equation (2.2), the marginal density function of  
 2  $X_{i:m:n:s}^{\mathbf{R}}$  is obtained as

$$f_{X_{i:m:n:s}^{\mathbf{R}}}(x_i|\alpha, \beta) = c_{i-1} s \alpha \beta x_i^{\beta-1} \sum_{j=1}^i a_j \exp\{-s \alpha \gamma_j x_i^{\beta}\}, \quad x_i > 0. \quad (2.3)$$

3 In the following subsections, two cases for the parameter  $\beta$  are considered.

### 5 2.1. $\beta$ is known

6 Assume a gamma conjugate prior for  $\alpha$  as

$$\pi(\alpha) \propto \alpha^{a-1} \exp\{-b\alpha\}, \quad \alpha > 0, \quad (2.4)$$

7 where  $a$  and  $b$  are positive hyperparameters that are chosen, for example, from a prior knowledge  
 8 of the mean and variance of  $\alpha$ . From (2.1) and (2.4), the posterior density function of  $\alpha$ , given  $\mathbf{u}$ ,  
 9 is obtained as

$$\pi(\alpha|\mathbf{u}) \propto \alpha^{r+a-1} \exp\{-\alpha g_{\beta,b}(u_r)\}, \quad \alpha > 0, \quad (2.5)$$

where  $g_{\beta,b}(u_r) = b + k u_r^{\beta}$ . It is obvious that  $(\alpha|\mathbf{u}) \sim \Gamma(r+a, g_{\beta,b}(u_r))$ . By combining the posterior  
 density function in (2.5) with the conditional density function in (2.3), and then integrating out  
 the parameter  $\alpha$ , the Bayes predictive density function of  $X_{i:m:n:s}^{\mathbf{R}}$ , given  $\mathbf{u}$ , is

$$\begin{aligned} f_{X_{i:m:n:s}^{\mathbf{R}}}^*(x_i|\mathbf{u}) &= \int_0^{\infty} f_{X_{i:m:n:s}^{\mathbf{R}}}(x_i|\alpha) \pi(\alpha|\mathbf{u}) d\alpha \\ &= c_{i-1} s \beta (r+a) (g_{\beta,b}(u_r))^{r+a} x_i^{\beta-1} \sum_{j=1}^i a_j (g_{\beta,b}(u_r) + s \gamma_j x_i^{\beta})^{-(r+a+1)}. \end{aligned} \quad (2.6)$$

From (2.6), the predictive survival function of  $X_{i:m:n:s}^{\mathbf{R}}$  given  $\mathbf{u}$  becomes

$$\begin{aligned} P(X_{i:m:n:s}^{\mathbf{R}} > t|\mathbf{u}) &= \int_t^{\infty} f_{X_{i:m:n:s}^{\mathbf{R}}}^*(x_i|\mathbf{u}) dx_i \\ &= c_{i-1} \sum_{j=1}^i \frac{a_j}{\gamma_j} \left( \frac{g_{\beta,b}(u_r)}{g_{\beta,b}(u_r) + s \gamma_j t^{\beta}} \right)^{r+a}. \end{aligned} \quad (2.7)$$

10 Suppose that  $\psi_i^*(\tau)$  is the  $\tau$ -th upper quantile of the predictive distribution of  $X_{i:m:n:s}^{\mathbf{R}}$ , i.e.  
 11  $\psi_i^*(\tau)$  satisfies  $P(X_{i:m:n:s}^{\mathbf{R}} > \psi_i^*(\tau)|\mathbf{u}) = \tau$ . Hence, it is clear that the values  $\psi_i^*(\tau)$  and  $\psi_i^*(1-\tau)$   
 12 are the (one-sided) upper and lower 100(1- $\tau$ )% Bayesian prediction bounds for  $X_{i:m:n:s}^{\mathbf{R}}$ . Also,  
 13 the two-sided equi-tailed 100(1- $\tau$ )% Bayesian prediction interval for  $X_{i:m:n:s}^{\mathbf{R}}$  is obtained as  
 14  $(\psi_i^*(1-\tau/2), \psi_i^*(\tau/2))$ . Although the quantiles of the predictive density function in Equation (2.7)  
 15 are not expressible in closed form, we can calculate them numerically. In Section 4, we have used  
 16 the mathematical package of *Mathematica* version 6 in order to obtain these quantiles.

REMARK 1. A closed form for the Bayesian prediction interval of  $X_{1:m:n:s}^{\mathbf{R}}$  is derived. The value  
 of  $X_{1:m:n:s}^{\mathbf{R}}$  will be the first failure time of the future sample of size  $N = n \times s$ . Putting  $i = 1$  in  
 Equations (2.6) and (2.7), we obtain

$$f_{X_{1:m:n:s}^{\mathbf{R}}}^*(x_1|\mathbf{u}) = n s \beta (r+a) (g_{\beta,b}(u_r))^{r+a} x_1^{\beta-1} (g_{\beta,b}(u_r) + n s x_1^{\beta})^{-(r+a+1)},$$

and

$$P(X_{1:m:n:s}^{\mathbf{R}} > t|\mathbf{u}) = \left( \frac{g_{\beta,b}(u_r)}{g_{\beta,b}(u_r) + n s t^{\beta}} \right)^{r+a}. \quad (2.8)$$

From (2.8), the  $\tau$ -th upper quantile is obtained as

$$\psi_1^*(\tau) = \left\{ \frac{g_{\beta,b}(u_r)}{ns} \left( \tau^{-\frac{1}{r+a}} - 1 \right) \right\}^{\frac{1}{\beta}}.$$

1 Thus, the two-sided equi-tailed  $100(1 - \tau)\%$  Bayes prediction interval for  $X_{1:m:n:s}^{\mathbf{R}}$  is

$$\left( \left\{ \frac{g_{\beta,b}(U_{r(k)})}{ns} \left( \left(1 - \frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right) \right\}^{\frac{1}{\beta}}, \left\{ \frac{g_{\beta,b}(U_{r(k)})}{ns} \left( \left(\frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right) \right\}^{\frac{1}{\beta}} \right). \quad (2.9)$$

Since, the marginal density function of  $U_{r(k)}$  under the Weibull distribution is obtained as (see, for example, Arnold *et al.* [5])

$$f_{U_{r(k)}}(u) = \frac{(k\alpha)^r \beta}{\Gamma(r)} u^{r\beta-1} \exp\{-k\alpha u^\beta\}, \quad u > 0,$$

where  $\Gamma(\cdot)$  denotes the complete gamma function, one can readily derive that  $U_{r(k)}^\beta \sim \Gamma(r, k\alpha)$ . Hence, the expected width ( $W$ ) of Bayesian prediction interval in Equation (2.9) becomes

$$\begin{aligned} E(W) &= \xi E \left( \frac{g_{\beta,b}(U_{r(k)})}{ns} \right)^{\frac{1}{\beta}} \\ &= \frac{\xi(k\alpha)^r \beta}{(ns)^{\frac{1}{\beta}} \Gamma(r)} \int_0^\infty (b + ku^\beta)^{\frac{1}{\beta}} u^{r\beta-1} \exp(-k\alpha u^\beta) du \\ &= \frac{\xi \exp(\alpha b)}{(ns\alpha)^{\frac{1}{\beta}} \Gamma(r)} \int_{\alpha b}^\infty y^{\frac{1}{\beta}} (y - \alpha b)^{r-1} \exp(-y) dy \\ &= \frac{\xi \exp(\alpha b)}{(ns\alpha)^{\frac{1}{\beta}} \Gamma(r)} \sum_{j=1}^{r-1} \binom{r-1}{j} (-\alpha b)^{r-1-j} \int_{\alpha b}^\infty y^{\frac{1}{\beta}+j} \exp(-y) dy \\ &= \frac{\xi \exp(\alpha b)}{(ns\alpha)^{\frac{1}{\beta}} \Gamma(r)} \sum_{j=1}^{r-1} \binom{r-1}{j} (-\alpha b)^{r-1-j} \Gamma\left(\frac{1}{\beta} + j + 1, \alpha b\right), \end{aligned} \quad (2.10)$$

where

$$\xi = \left( \left(\frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right)^{\frac{1}{\beta}} - \left( \left(1 - \frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right)^{\frac{1}{\beta}},$$

2 and  $\Gamma(\cdot, \cdot)$  denotes the incomplete gamma function. In the special case when  $b = 0$ , the expression in  
 3 (2.10) simply becomes  $E(W) = \xi \Gamma(1/\beta + r) / ((ns\alpha)^{1/\beta} \Gamma(r))$ . From Equation (2.10), it is observed  
 4 that the expected width of Bayesian prediction interval for  $X_{1:m:n:s}^{\mathbf{R}}$  is decreasing with respect to  $n$   
 5 and  $s$  when all others are fixed.

6 It is easy to show that  $W = 1 + ns(X_{1:m:n:s}^{\mathbf{R}})^\beta / g_{\beta,b}(u_r)$  has the Pareto distribution with the  
 7 density function  $f_W(w) = (r+a)w^{-(r+a+1)}$ ,  $w > 1$ . Thus, we can find Bayesian prediction intervals  
 8 for  $X_{1:m:n:s}^{\mathbf{R}}$  on the basis of pivotal quantity  $W$ . One can readily obtain, in this case, the same  
 9 Bayesian prediction bounds as in Equation (2.9).

## 2.2. $\alpha$ and $\beta$ are unknown

In order to obtain Bayesian prediction interval for  $X_{i:m:n:s}^{\mathbf{R}}$ , specifying a general joint prior for  $\alpha$  and  $\beta$  leads us to computational complexities. For sake of brevity in the Bayesian analysis, we use Soland's method (see, Soland [26]). Therefore, it is assumed that the shape parameter  $\beta$  is restricted to a finite number of values  $\beta_1, \dots, \beta_M$  with prior probabilities  $\xi_1, \dots, \xi_M$ , respectively, such that

$0 < \xi_l < 1$  for  $l = 1, \dots, M$  and  $\sum_{l=1}^M \xi_l = 1$ . Moreover, suppose that the conditional distribution of  $\alpha$ , given  $\beta = \beta_l$ , is gamma with shape and scale parameters  $a_l$  and  $b_l$ , respectively. Therefore,

$$P(\beta = \beta_l) = \xi_l, \quad l = 1, \dots, M,$$

$$\pi(\alpha|\beta = \beta_l) = \frac{b_l^{a_l}}{\Gamma(a_l)} \alpha^{a_l-1} \exp(-b_l \alpha), \quad \alpha > 0, a_l > 0, b_l > 0.$$

The hyperparameters  $a_l$  and  $b_l$  ( $l = 1, \dots, M$ ) are chosen so as to reflect prior beliefs on  $\alpha$  given that  $\beta = \beta_l$ . The conditional prior density function of  $\alpha$  given  $\beta = \beta_l$  for  $l = 1, \dots, M$  is completely specified when the hyperparameters  $a_l$  and  $b_l$  are known. To simplify the problem, we assume that the shape and scale parameters of the conditional distribution  $\alpha$  given  $\beta = \beta_l$  are  $a$  and  $b\beta_l$ , respectively, where  $a$  and  $b$  are two positive specified constants. Thus, using the LF (2.1), the conditional posterior density function of  $\alpha$  given  $\beta = \beta_l$  is

$$\pi(\alpha|\beta = \beta_l, \mathbf{u}) = \frac{L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)}{\int_0^\infty L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)d\alpha} \propto \alpha^{r+a-1} \exp(-\alpha q_{\beta_l, b}(u_r)), \quad (2.11)$$

where  $q_{\beta_l, b}(u_r) = b\beta_l + ku_r^{\beta_l}$ . Consequently,  $(\alpha|\beta = \beta_l, \mathbf{u}) \sim \Gamma(r + a, q_{\beta_l, b}(u_r))$  for  $l = 1, \dots, M$ . Also, the posterior density function of  $\beta = \beta_l$  is obtained as

$$P(\beta = \beta_l|\mathbf{u}) = \frac{\int_0^\infty L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)d\alpha}{\sum_{l=1}^M \int_0^\infty L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)d\alpha} = c^* \xi_l e_{\beta_l}(\mathbf{u}) \left( \frac{\beta_l}{q_{\beta_l, b}(u_r)} \right)^{r+a}, \quad (2.12)$$

1 where  $e_{\beta_l}(\mathbf{u}) = \prod_{j=1}^r u_j^{\beta_l-1}$  and  $c^* = \left[ \sum_{l=1}^M \xi_l e_{\beta_l}(\mathbf{u}) \left\{ \beta_l / q_{\beta_l, b}(u_r) \right\}^{r+a} \right]^{-1}$  is the normalizing constant.

2 The Bayes predictive density function of  $X_{i:m:n:s}^{\mathbf{R}}$ , given  $\mathbf{u}$ , is

$$f_{X_{i:m:n:s}^{\mathbf{R}}}^{**}(x_i|\mathbf{u}) = \sum_{l=1}^M \int_0^\infty f_{X_{i:m:n:s}^{\mathbf{R}}}(x_i|\alpha, \beta_l)\pi(\alpha|\beta = \beta_l, \mathbf{u})P(\beta = \beta_l|\mathbf{u})d\alpha. \quad (2.13)$$

3 Substituting Equations (2.3), (2.11) and (2.12) into Equation (2.13), the Bayes predictive density  
4 function of  $X_{i:m:n:s}^{\mathbf{R}}$  reads

$$f_{X_{i:m:n:s}^{\mathbf{R}}}^{**}(x_i|\mathbf{u}) = c^* c_{i-1} s(r+a) \sum_{l=1}^M \sum_{j=1}^i \xi_l a_j e_{\beta_l}(\mathbf{u}) x_i^{\beta_l-1} \left( \frac{\beta_l}{q_{\beta_l, b}(u_r) + s\gamma_j x_i^{\beta_l}} \right)^{r+a+1}, \quad (2.14)$$

and hence, the predictive survival function of  $X_{i:m:n:s}^{\mathbf{R}}$  becomes

$$P(X_{i:m:n:s}^{\mathbf{R}} \geq t|\mathbf{u}) = \int_t^\infty f_{X_{i:m:n:s}^{\mathbf{R}}}^{**}(x_i|\mathbf{u})dx_i = c^* c_{i-1} \sum_{l=1}^M \sum_{j=1}^i \xi_l a_j \gamma_j^{-1} e_{\beta_l}(\mathbf{u}) \left( \frac{\beta_l}{q_{\beta_l, b}(u_r) + s\gamma_j t^{\beta_l}} \right)^{r+a}. \quad (2.15)$$

5 If  $\psi_i^{**}(\tau)$  satisfies  $P(X_{i:m:n:s}^{\mathbf{R}} > \psi_i^{**}(\tau)|\mathbf{u}) = \tau$ , hence, in this case, the values  $\psi_i^{**}(\tau)$  and  $\psi_i^{**}(1 -$   
6  $\tau)$  are the (one-sided) upper and lower  $100(1 - \tau)\%$  Bayesian prediction bounds for  $X_{i:m:n:s}^{\mathbf{R}}$ .

Also, the two-sided equi-tailed  $100(1 - \tau)\%$  Bayesian prediction interval for  $X_{i:m:n:s}^{\mathbf{R}}$  is given by  $(\psi_i^{**}(1 - \tau/2), \psi_i^{**}(\tau/2))$ . Again, the quantiles of the predictive density function in Equation (2.15) are not expressible in closed form and then we calculate them numerically using the mathematical package of *Mathematica* version 6 in Section 4.

It is important to determine the values for the hyperparameters  $(\beta_l, \xi_l), l = 1, \dots, M$  and  $(a, b)$ . In fact, considering the values for  $(\beta_l, \xi_l), l = 1, \dots, M$  is fairly straightforward, but for  $(a, b)$ , the following method is suggested. Suppose that on the basis of a prior knowledge, we known the conditional expectation and variance of  $\alpha$  given a special value of  $\beta$  (say,  $\beta = \beta_t$ ). Consequently, we have

$$\mu_t = E(\alpha|\beta = \beta_t) = \frac{a}{b\beta_t} \quad \text{and} \quad \sigma_t^2 = Var(\alpha|\beta = \beta_t) = \frac{a}{(b\beta_t)^2}, \quad (2.16)$$

where  $\mu_t$  and  $\sigma_t^2$  are the expectation and variance of  $\alpha$  given  $\beta = \beta_t$ , respectively. By solving Equation (2.16), the hyperparameters  $(a, b)$  are obtained.

REMARK 2. When  $i = 1$ , the expressions in Equations (2.14) and (2.15) are reduced to

$$f_{X_{1:m:n:s}^{\mathbf{R}}}^{**}(x_1|\mathbf{u}) = c^*ns(r+a) \sum_{l=1}^M \xi_l e_{\beta_l}(\mathbf{u}) x_1^{\beta_l-1} \left( \frac{\beta_l}{q_{\beta_l,b}(u_r) + nsx_1^{\beta_l}} \right)^{-(r+a+1)},$$

and

$$P(X_{1:m:n:s}^{\mathbf{R}} \geq t|\mathbf{u}) = c^* \sum_{l=1}^M \xi_l e_{\beta_l}(\mathbf{u}) \left( \frac{\beta_l}{q_{\beta_l,b}(u_r) + nst^{\beta_l}} \right)^{r+a}, \quad (2.17)$$

respectively. Thus, the  $100(1 - \tau)\%$  Bayesian prediction interval  $(\psi_1^{**}(1 - \tau/2), \psi_1^{**}(\tau/2))$  for  $X_{1:m:n:s}^{\mathbf{R}}$  can be easily obtained numerically using Equation (2.17).

### 3. Bayes point prediction

In the Bayesian framework, the choice of a loss function is essential. One of the most popular widely used loss functions is the squared error (SE) loss defined by  $L(\hat{\delta}, \delta) = (\hat{\delta} - \delta)^2$ , where  $\hat{\delta}$  is an estimate of  $\delta$ . But, the SE loss function is justified only when losses are symmetric in nature. The symmetric nature of this loss function gives equal weight to overestimation as well as underestimation, while in practice, overestimation may be more serious than underestimation of same magnitude or vice versa. Such conditions are very common in engineering, medical and biomedical sciences. In this case, an asymmetric loss function might be more appropriate. A suitable alternative to the SE loss function is a convex but asymmetric loss function, known as the LINear-EXponential (LINEX) loss function, proposed by Varian [29] and defined by

$$L(\hat{\delta}, \delta) = \exp\{\vartheta(\hat{\delta} - \delta)\} - \vartheta(\hat{\delta} - \delta) - 1, \quad \vartheta \neq 0. \quad (3.1)$$

The constants  $\vartheta$  and  $\sigma > 0$  involved in (3.1) are the shape and scale parameters. Obviously, the nature of the LINEX loss function changes according to the choice of  $\vartheta$ . The sign and magnitude of the shape parameter  $\vartheta$  represents the direction and degree of symmetry, respectively. ( $\vartheta > 0$  means overestimation is more serious than underestimation and  $\vartheta < 0$  means the opposite). The LINEX loss converges to the SE loss as  $\vartheta \rightarrow 0$ . It is proved that the Bayes estimates of  $\delta$  under the SE and the LINEX loss functions, respectively denoted by  $\hat{\delta}_{BS}$  and  $\hat{\delta}_{BL}$ , are given by

$$\hat{\delta}_{BS} = E(\delta), \quad (3.2)$$

and

$$\hat{\delta}_{BL} = -\frac{1}{\vartheta} \ln E(\exp(-\vartheta\delta)), \quad (3.3)$$

provided that  $E(\delta)$  and  $E(\exp(-\vartheta\delta))$  exist and are finite.

Now, we obtain the Bayes point predictor for  $X_{i:m:n:s}^{\mathbf{R}}$  on the basis of observed  $k$ -record data. For this purpose, we consider the following two cases.

### 3.1. $\beta$ is known

Using Equations (2.6), (3.2) and (3.3), the Bayes point predictors for  $X_{i:m:n:s}^{\mathbf{R}}$  under SE and LINEX loss functions are given, respectively, by

$$\begin{aligned}\hat{X}_{i:m:n:s,BS}^{\mathbf{R}} &= \int_0^\infty x_i f_{X_{i:m:n:s}^{\mathbf{R}}}^*(x_i|\mathbf{u}) dx_i \\ &= c_{i-1}(r+a) \sum_{j=1}^i \frac{a_j}{\gamma_j} \left( \frac{s\gamma_j}{g_{\beta,b}(u_r)} \right)^{-\frac{1}{\beta}} \int_0^1 y^{r+a-\frac{1}{\beta}-1} (1-y)^{\frac{1}{\beta}} dy \\ &= c_{i-1}(r+a) B\left(r+a-\frac{1}{\beta}, 1+\frac{1}{\beta}\right) \sum_{j=1}^i \frac{a_j}{\gamma_j} \left( \frac{s\gamma_j}{g_{\beta,b}(u_r)} \right)^{-\frac{1}{\beta}},\end{aligned}\quad (3.4)$$

and

$$\begin{aligned}\hat{X}_{i:m:n:s,BL}^{\mathbf{R}} &= -\frac{1}{\vartheta} \ln \left\{ \int_0^\infty \exp(-\vartheta x_i) f_{X_{i:m:n:s}^{\mathbf{R}}}^*(x_i|\mathbf{u}) dx_i \right\} \\ &= -\frac{1}{\vartheta} \ln \left\{ c_{i-1} s \beta (r+a) (g_{\beta,b}(u_r))^{r+a} \sum_{j=1}^i a_j \int_0^\infty \frac{x_i^{\beta-1} \exp(-\vartheta x_i)}{(g_{\beta,b}(u_r) + s\gamma_j x_i^\beta)^{r+a+1}} dx_i \right\} \\ &= -\frac{1}{\vartheta} \ln \left\{ c_{i-1} s \beta (r+a) \sum_{j=1}^i \sum_{l=0}^\infty \binom{-r-a-1}{l} \frac{a_j (s\gamma_j)^l \Gamma((l+1)\beta)}{(g_{\beta,b}(u_r))^{l+1} \vartheta^{(l+1)\beta}} \right\},\end{aligned}\quad (3.5)$$

1 where  $B(\cdot, \cdot)$  denotes the complete beta function.

2 REMARK 3. In the special case  $i = 1$ , the expressions in Equations (3.4) and (3.5) simply  
 3 become

$$\hat{X}_{1:m:n:s,BS}^{\mathbf{R}} = (r+a) B\left(r+a-\frac{1}{\beta}, 1+\frac{1}{\beta}\right) \left( \frac{ns}{g_{\beta,b}(u_r)} \right)^{-\frac{1}{\beta}},$$

and

$$\hat{X}_{1:m:n:s,BL}^{\mathbf{R}} = -\frac{1}{\vartheta} \ln \left\{ ns \beta (r+a) \sum_{l=0}^\infty \binom{-r-a-1}{l} \frac{(ns)^l \Gamma((l+1)\beta)}{(g_{\beta,b}(u_r))^{l+1} \vartheta^{(l+1)\beta}} \right\},$$

4 respectively.

### 3.2. $\alpha$ and $\beta$ are unknown

From Equations (2.14), (3.2) and (3.3), it is easy to show that the Bayes point predictors for  $X_{i:m:n:s}^{\mathbf{R}}$  under SE and LINEX loss functions are, respectively,

$$\begin{aligned}\hat{X}_{i:m:n:s,BS}^{\mathbf{R}} &= c^* c_{i-1} s (r+a) \\ &\quad \times \sum_{l=1}^M \sum_{j=1}^i \xi_l \beta_l^{r+a} a_j e_{\beta_l}(\mathbf{u}) B\left(r+a-\frac{1}{\beta_l}, 1+\frac{1}{\beta_l}\right) \frac{(s\gamma_j)^{-\frac{1}{\beta_l}-1}}{(g_{\beta_l,b}(u_r))^{r+a-\frac{1}{\beta_l}}},\end{aligned}\quad (3.6)$$

and

$$\begin{aligned}\hat{X}_{i:m:n:s,BL}^{\mathbf{R}} &= -\frac{1}{\vartheta} \ln \left\{ c^* c_{i-1} s (r+a) \right. \\ &\quad \left. \times \sum_{l=1}^M \sum_{j=1}^i \sum_{h=0}^\infty \xi_l \beta_l^{r+a+1} a_j e_{\beta_l}(\mathbf{u}) \binom{-r-a-1}{h} \frac{(s\gamma_j)^h \Gamma((h+1)\beta_l)}{(g_{\beta_l,b}(u_r))^{r+a+h+1} \vartheta^{(h+1)\beta_l}} \right\}.\end{aligned}\quad (3.7)$$

REMARK 4. For the special case of  $i = 1$ , the expressions in Equations (3.6) and (3.7) are reduced to

$$\hat{X}_{1:m:n:s,BS}^R = c^* ns(r+a) \sum_{l=1}^M \xi_l \beta_l^{r+a} e_{\beta_l}(\mathbf{u}) B\left(r+a-\frac{1}{\beta_l}, 1+\frac{1}{\beta_l}\right) \frac{(ns)^{-\frac{1}{\beta_l}-1}}{(g_{\beta_l,b}(u_r))^{r+a-\frac{1}{\beta_l}}},$$

and

$$\hat{X}_{1:m:n:s,BL}^R = -\frac{1}{\vartheta} \ln \left\{ c^* ns(r+a) \times \sum_{l=1}^M \sum_{h=0}^{\infty} \xi_l \beta_l^{r+a+1} e_{\beta_l}(\mathbf{u}) \binom{-r-a-1}{h} \frac{(ns)^h \Gamma((h+1)\beta_l)}{(g_{\beta_l,b}(u_r))^{r+a+h+1} \vartheta^{(h+1)\beta_l}} \right\},$$

1 respectively.

## 2 4. Numerical Results

3 In this section, the performances of the procedures proposed in the preceding sections are investigated by conducting a simulation study and analysing an illustrative example.

### 5 4.1. Simulation study

6 A simulation study was conducted in order to evaluate the performance of the Bayesian prediction  
 7 intervals and Bayes point predictors for  $i$ -th order statistic  $X_{i:m:n:s}^R$  in a future progressively first-  
 8 failure-censored sample under the assumption that  $\alpha$  and  $\beta$  are unknown. Since in predicting first  
 9 order statistic  $X_{1:m:n:s}^R$ , the corresponding Bayesian prediction intervals and point predictors depend  
 10 on the total number of units put on the test  $N = n \times s$  but not on  $(R_1, \dots, R_m)$  and  $m$ , hence,  
 11 without loss of generality, we constructed the Bayesian prediction intervals and point predictors  
 12 for  $X_{1:m:n:s}^R$  under the assumption that  $m = N$  and  $(R_1, \dots, R_m) = (0, \dots, 0)$ . Also, for predicting  
 13  $X_{5:m:n:s}^R$ , we considered different choices for progressive censoring scheme  $(R_1, \dots, R_m)$  which include  
 14 three progressive censoring schemes  $(n-m, 0_{(m-1)})$ ,  $(0_{(m-1)}, n-m)$  and  $(0_{(3)}, n-m, 0_{(m-4)})$ , among  
 15 others. Note that, for example, the notation  $(15, 0_{(4)})$  stands for a progressive censoring scheme  
 16 with specification  $(R_1, \dots, R_5) = (15, 0, 0, 0, 0)$ .

17 Our simulation was done according to the following steps:

18 1. Assuming  $\beta_t = 1$  and  $(\mu_t, \sigma_t^2) = (1, 0.1)$  and using Equation (2.16), the hyperparameters  $(a, b)$   
 19 were obtained as  $(10, 10)$ .

20 2. Following the work of Soman and Misra [28], we assumed that the value of the shape parameter  
 21  $\beta$  falls in the interval  $(0, 3)$ . Then, we generated  $\beta = 2.5$  from discrete uniform distribution on  
 22  $0.5(0.5)3$  and  $\alpha = 0.27152$  from  $\Gamma(a, b\beta)$  prior distribution.

23 3. A sequence of independent observations from Weibull distribution with parameters  $\alpha =$   
 24  $0.27152$  and  $\beta = 2.5$  was generated and the first  $r$   $k$ -record values in the sequence for  $r = 5$  and  
 25  $k = 1, 3$  were discovered.

26 4. Considering the observed  $k$ -record values and using Equations (2.15), (3.6) and (3.7), the  
 27 Bayesian two-sided equi-tailed prediction intervals as well as the Bayes point predictors for  $X_{i:m:n:s}^R$   
 28 ( $i = 1, 5$ ) under the SE and LINEX loss functions were obtained.

5. Simulating  $t = 10^4$  independent progressively first-failure-censored samples, the coverage probabilities of the Bayesian prediction intervals for  $X_{i:m:n:s}^R$  ( $i = 1, 5$ ) were computed. Also, the Estimated Risks (ER) of the Bayes point predictors  $X_{i:m:n:s}^R$  under the SE and LINEX loss functions given, respectively, by

$$\text{ER}(\hat{\delta}_{BS}) = \frac{1}{t} \sum_{j=1}^t \left( \hat{\delta}_{BS} - \delta_j \right)^2,$$

TABLE 3. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for  $X_{i:m:n:s}^{\mathbf{R}}$  with  $i = 1$  and  $k = 1$ .

N	Coverage probability		Estimated risk		
	$\tau = 0.1$	$\tau = 0.05$	SE	LINEX	
				$\vartheta = 0.7$	$\vartheta = 1.2$
5	0.8956	0.9456	0.13348	0.03170	0.09100
10	0.8929	0.9514	0.07721	0.01857	0.05378
15	0.8942	0.9506	0.05550	0.01345	0.03917
20	0.9054	0.9547	0.04317	0.01050	0.03068
30	0.9038	0.9498	0.03160	0.00771	0.02257
50	0.8910	0.9534	0.02145	0.00525	0.01541

TABLE 4. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for  $X_{i:m:n:s}^{\mathbf{R}}$  with  $i = 1$  and  $k = 3$ .

N	Coverage probability		Estimated risk		
	$\tau = 0.1$	$\tau = 0.05$	SE	LINEX	
				$\vartheta = 0.7$	$\vartheta = 1.2$
5	0.9101	0.9581	0.13041	0.03125	0.09020
10	0.9082	0.9547	0.07815	0.01893	0.05509
15	0.8924	0.9430	0.05943	0.01443	0.04206
20	0.9076	0.9608	0.04701	0.01145	0.03348
30	0.8929	0.9429	0.03396	0.00832	0.02441
50	0.9058	0.9582	0.02343	0.00576	0.01694

and

$$\text{ER}(\hat{\delta}_{BL}) = \frac{1}{t} \sum_{j=1}^t \left( \exp \left\{ \vartheta (\hat{\delta}_{BL} - \delta_j) \right\} - \vartheta (\hat{\delta}_{BL} - \delta_j) - 1 \right),$$

1 were calculated.

2 The results are tabulated in Tables 3-6. From empirical evidence in Tables 3-6, the following points  
3 can be drawn:

4 • The simulated coverage probabilities of the Bayesian prediction intervals for  $X_{i:m:n:s}^{\mathbf{R}}$  are close  
5 to the their nominal level.

6 • The estimated risk of predictors  $\hat{X}_{1:m:n:s,BS}^{\mathbf{R}}$  and  $\hat{X}_{1:m:n:s,BL}^{\mathbf{R}}$  is decreasing in  $N$ .

7 • For fixed  $s$  and  $n$ , as  $m$  increases, the estimated risk of predictors  $\hat{X}_{5:m:n:s,BS}^{\mathbf{R}}$  and  $\hat{X}_{5:m:n:s,BL}^{\mathbf{R}}$   
8 reduce significantly under progressive censoring schemes  $(n - m, 0_{(m-1)})$  and  $(0_{(3)}, n - m, 0_{(m-4)})$ .

9 • It seems that the estimated risk of  $\hat{X}_{5:m:n:s,BS}^{\mathbf{R}}$  and  $\hat{X}_{5:m:n:s,BL}^{\mathbf{R}}$  under progressive censoring  
10 scheme  $(0_{(m-1)}, n - m)$  does not depend on  $m$ , when all other components are kept fixed.

11 • For fixed  $s$  (or  $n$ ),  $m$  and progressive censoring scheme, when  $n$  (or  $s$ ) becomes large, the  
12 estimated risk of  $\hat{X}_{5:m:n:s,BS}^{\mathbf{R}}$  and  $\hat{X}_{5:m:n:s,BL}^{\mathbf{R}}$  decreases.

13 • When  $s$ ,  $n$  and  $m$  are fixed, the progressive censoring scheme  $(0_{(m-1)}, n - m)$  possesses the  
14 smallest estimated risk. Also, it seems that when  $s$  and  $n$  are fixed, the distance between the  
15 estimated risk of different progressive censoring schemes decreases with respect to  $m$ .

16 • Experimentation with different values of the prior parameters led to the same results, which  
17 shows a good stability with respect to the prior setting. The results are available upon readers  
18 request to the authors.

## 4.2. Illustrative example

Dunsmore [12] has given the size of rock crushed by a rock crushing machine. The machine has to be reset if, at any operation, the size of rock being crushed is larger than that has been crushed

TABLE 5. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for  $X_{i:m:n:s}^R$  with  $i = 5$  and  $k = 1$ .

s	n	m	$(R_1, \dots, R_m)$	Coverage probability		Estimated risk			
				$\tau = 0.1$	$\tau = 0.05$	SE	LINEX		
							$\vartheta = 0.7$	$\vartheta = 1.2$	
1	20	5	$(15, 0_{(4)})$	0.9066	0.9664	0.32170	0.07763	0.22375	
			$(0_{(4)}, 15)$	0.8750	0.9382	0.06308	0.01601	0.04828	
			$(0_{(3)}, 15, 0)$	0.8952	0.9517	0.35890	0.08169	0.22792	
			$(3_{(5)})$	0.8732	0.9314	0.11942	0.02938	0.08675	
			$(0, 5_{(3)}, 0)$	0.9037	0.9589	0.32485	0.07538	0.21285	
	15	5	$(5, 0_{(14)})$	0.8723	0.9326	0.07678	0.01937	0.05816	
			$(0_{(14)}, 5)$	0.8785	0.9426	0.06297	0.01600	0.04824	
			$(0_{(3)}, 5, 0_{(11)})$	0.8815	0.9414	0.06828	0.01730	0.05206	
			$(1, 0_{(2)}, 1, 0_{(2)}, \dots, 1, 0_{(2)})$	0.8762	0.9405	0.06606	0.01676	0.05051	
	50	15	$(35, 0_{(14)})$	0.8718	0.9351	0.07639	0.01926	0.05780	
			$(0_{(14)}, 35)$	0.9241	0.9692	0.03005	0.00775	0.02361	
			$(0_{(3)}, 35, 0_{(11)})$	0.8939	0.9440	0.05537	0.01383	0.04122	
			$(0, 5, 0, 5, \dots, 0, 5, 0)$	0.9229	0.9651	0.03280	0.00845	0.02570	
		30	5	$(20, 0_{(29)})$	0.8989	0.9545	0.04395	0.01123	0.03400
				$(0_{(29)}, 20)$	0.9257	0.9693	0.02992	0.00772	0.02352
				$(0_{(3)}, 20, 0_{(26)})$	0.9157	0.9604	0.03485	0.00893	0.02708
$(0, 1_{(2)}, 0, 1_{(2)}, \dots, 0, 1_{(2)})$				0.9241	0.9687	0.03105	0.00800	0.02434	
50	5	$(5, 0_{(28)}, 15)$	0.9175	0.9660	0.03283	0.00844	0.02567		
		$(2, 0_{(2)}, 2, 0_{(2)}, \dots, 2, 0_{(2)})$	0.9218	0.9654	0.03185	0.00820	0.02493		
		$(15, 0_{(4)})$	0.8786	0.9421	0.15401	0.03794	0.11205		
		$(0_{(4)}, 15)$	0.9199	0.9683	0.03383	0.00872	0.02652		
3	20	5	$(0_{(3)}, 15, 0)$	0.8893	0.9363	0.16443	0.03884	0.11144	
			$(3_{(5)})$	0.8851	0.9428	0.06340	0.01598	0.04788	
			$(0, 5_{(3)}, 0)$	0.8874	0.9404	0.15552	0.03726	0.10781	
			$(5, 0_{(14)})$	0.9079	0.9601	0.04133	0.01063	0.03227	
			$(0_{(14)}, 5)$	0.9212	0.9704	0.03373	0.00871	0.02653	
	15	5	$(0_{(3)}, 5, 0_{(11)})$	0.9163	0.9638	0.03660	0.00942	0.02860	
			$(1, 0_{(2)}, 1, 0_{(2)}, \dots, 1, 0_{(2)})$	0.9167	0.9649	0.03571	0.00920	0.02798	
			$(35, 0_{(14)})$	0.9053	0.9606	0.04123	0.01056	0.03196	
			$(0_{(14)}, 35)$	0.9257	0.9791	0.01651	0.00427	0.01301	
	50	15	$(0_{(3)}, 35, 0_{(11)})$	0.8985	0.9569	0.02754	0.00699	0.02106	
			$(0, 5, 0, 5, \dots, 0, 5, 0)$	0.9223	0.9791	0.01771	0.00458	0.01396	
			$(20, 0_{(29)})$	0.9079	0.9704	0.02350	0.00607	0.01846	
			$(0_{(29)}, 20)$	0.9243	0.9778	0.01668	0.00431	0.01313	
		30	5	$(0_{(3)}, 20, 0_{(26)})$	0.9189	0.9768	0.01854	0.00479	0.01458
				$(0, 1_{(2)}, 0, 1_{(2)}, \dots, 0, 1_{(2)})$	0.9250	0.9804	0.01654	0.00428	0.01305
				$(5, 0_{(28)}, 15)$	0.9237	0.9792	0.01737	0.00450	0.01372
$(2, 0_{(2)}, 2, 0_{(2)}, \dots, 2, 0_{(2)})$				0.9286	0.9815	0.01690	0.00437	0.01333	

before. The following data are the sizes dealt with up to the third time that the machine has been reset.

9.3 0.6 24.4 18.1 6.6 9.0 14.3 6.6 13.0 2.4 5.6 33.8.

1 Dunsmore [12] assumed that the exponential distribution is suitable for these data. Here, we intend  
 2 to predict the progressively first-failure-censored order statistics from a future sample based on  
 3  $k$ -records extracted from the above data. The observed  $k$ -record values, when  $k = 1$  and  $k = 2$ , are  
 4 shown in Table 7. For this example, the various progressive first-failure-censoring schemes (CS)  
 5 reported in Table 8 were considered. We recall that in the case where  $\beta$  is known, the gamma  
 6 conjugate prior (2.4) was considered for  $\alpha$ , so we readily find  $E(\alpha) = a/b$  and  $Var(\alpha) = a/b^2$ . In  
 7 order to have least informative, informative, and most informative prior densities, we considered

TABLE 6. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for  $X_{i:m:n:s}^R$  with  $i = 5$  and  $k = 3$ .

s	n	m	$(R_1, \dots, R_m)$	Coverage probability		Estimated risk			
				$\tau = 0.1$	$\tau = 0.05$	SE	LINEX		
							$\vartheta = 0.7$	$\vartheta = 1.2$	
1	20	5	$(15, 0_{(4)})$	0.8909	0.9620	0.30663	0.07283	0.20736	
			$(0_{(4)}, 15)$	0.8897	0.9477	0.04746	0.01184	0.03530	
			$(0_{(3)}, 15, 0)$	0.8840	0.9470	0.33897	0.07651	0.21198	
			$(3_{(5)})$	0.8729	0.9371	0.10126	0.02447	0.07129	
			$(0, 5_{(3)}, 0)$	0.8896	0.9496	0.31201	0.07054	0.19593	
	15	15	$(5, 0_{(14)})$	0.8754	0.9362	0.06167	0.01527	0.04526	
			$(0_{(14)}, 5)$	0.8862	0.9435	0.04757	0.01185	0.03530	
			$(0_{(3)}, 5, 0_{(11)})$	0.8870	0.9435	0.05194	0.01289	0.03829	
			$(1, 0_{(2)}, 1, 0_{(2)}, \dots, 1, 0_{(2)})$	0.8834	0.9409	0.05067	0.01261	0.03748	
	50	15	$(35, 0_{(14)})$	0.8799	0.9376	0.06048	0.01495	0.04426	
			$(0_{(14)}, 35)$	0.9372	0.9723	0.02050	0.00519	0.01566	
			$(0_{(3)}, 35, 0_{(11)})$	0.9108	0.9515	0.04200	0.01029	0.03029	
			$(0, 5, 0, 5, \dots, 0, 5, 0)$	0.9302	0.9684	0.02298	0.00581	0.01747	
			30	$(20, 0_{(29)})$	0.9141	0.9587	0.03155	0.00793	0.02377
		$(0_{(29)}, 20)$		0.9396	0.9744	0.02033	0.00515	0.01554	
		$(0_{(3)}, 20, 0_{(26)})$		0.9307	0.9680	0.02417	0.00610	0.01834	
		$(0, 1_{(2)}, 0, 1_{(2)}, \dots, 0, 1_{(2)})$		0.9376	0.9733	0.02055	0.00521	0.01573	
		$(5, 0_{(28)}, 15)$		0.9336	0.9718	0.02193	0.00555	0.01675	
		3	20	5	$(15, 0_{(4)})$	0.8704	0.9337	0.13125	0.03141
	$(0_{(4)}, 15)$				0.9381	0.9727	0.01889	0.00479	0.01446
$(0_{(3)}, 15, 0)$	0.8928				0.9424	0.13666	0.03185	0.09047	
$(3_{(5)})$	0.9037				0.9499	0.04094	0.01013	0.03005	
$(0, 5_{(3)}, 0)$	0.8861				0.9387	0.13453	0.03138	0.08928	
15	15		$(5, 0_{(14)})$	0.9294	0.9666	0.02414	0.00610	0.01837	
			$(0_{(14)}, 5)$	0.9405	0.9739	0.01870	0.00474	0.01430	
			$(0_{(3)}, 5, 0_{(11)})$	0.9355	0.9708	0.02121	0.00536	0.01614	
			$(1, 0_{(2)}, 1, 0_{(2)}, \dots, 1, 0_{(2)})$	0.9375	0.9748	0.01970	0.00499	0.01505	
			50	15	$(35, 0_{(14)})$	0.9140	0.9700	0.02405	0.00606
$(0_{(14)}, 35)$	0.9374				0.9797	0.00793	0.00204	0.00615	
$(0_{(3)}, 35, 0_{(11)})$	0.9170				0.9641	0.01628	0.00406	0.01209	
$(0, 5, 0, 5, \dots, 0, 5, 0)$	0.9337				0.9775	0.00873	0.00223	0.00674	
30	$(20, 0_{(29)})$				0.9251	0.9777	0.01234	0.00314	0.00949
	$(0_{(29)}, 20)$		0.9372	0.9855	0.00810	0.00208	0.00626		
	$(0_{(3)}, 20, 0_{(26)})$	0.9313	0.9764	0.00947	0.00241	0.00727			
	$(0, 1_{(2)}, 0, 1_{(2)}, \dots, 0, 1_{(2)})$	0.9385	0.9798	0.00794	0.00203	0.00615			
	$(5, 0_{(28)}, 15)$	0.9389	0.9789	0.00855	0.00218	0.00661			
$(2, 0_{(2)}, 2, 0_{(2)}, \dots, 2, 0_{(2)})$	0.9379	0.9784	0.00822	0.00211	0.00635				

TABLE 7. Observed  $k$ -record values extracted from the data set in Dunsmore [12], when  $k = 1$  and  $k = 2$ .

$j$	1	2	3	4
$U_{j(1)}$	9.3	24.4	33.8	
$U_{j(2)}$	0.6	9.3	18.1	24.4

- 1 three cases for the hyperparameters  $(a, b) = (0.5, 0.5), (1, 1)$  and  $(2, 2)$ , which correspond to the
- 2 prior densities with common means 1 and different variances 2, 1 and 0.5, respectively. Considering
- 3 these settings, we constructed 95% Bayesian two-sided equi-tailed prediction intervals as well as
- 4 Bayes point predictors for future progressively first-failure-censored order statistics under SE and

TABLE 8. Different choices for progressive first-failure-censoring scheme.

CS	$s$	$n$	$m$	$(R_1, \dots, R_m)$
I	1	10	5	$(5, 0_{(4)})$
II	1	10	8	$(0_{(4)}, 2, 0_{(3)})$
III	2	10	10	$(0_{(10)})$
IV	2	30	10	$(0_{(5)}, 4_{(5)})$
V	3	30	10	$(1_{(3)}, 2_{(4)}, 3_{(3)})$
VI	3	30	20	$(0_{(3)}, 2, 0_{(3)}, 1, 1, 4, 0_{(3)}, 1, 0_{(3)}, 1, 0_{(2)})$

1 LINEX loss functions. The results are summarized in Tables 9-14.

2

3 By empirical evidences from Tables 9-14, we observe that the width of the prediction intervals  
 4 decreases as the information about the unknown parameter  $\alpha$  increases, i.e., as the hyperparameters  
 5  $(a, b)$  increase. So, the prediction intervals are shortest when  $(a, b) = (2, 2)$ . As we would expect, the  
 6 prediction intervals are very sensitive to the prior distribution, and hence, the prior distribution  
 7 must be determined carefully by the researcher. Also, the width of the prediction intervals is  
 8 increasing with respect to  $i$  when other parameters are kept fixed, while it decreases as each of  
 9 the parameters  $m, n$  and  $s$  increases. Moreover, the results in Tables 9-14 show that the Bayes  
 10 point predictor relative to the LINEX loss function is sensitive to the value of the shape parameter  
 11  $\vartheta$ . The problem of choosing the value of the parameter  $\vartheta$  has been investigated by Calabria and  
 12 Pulcini [10]. These results establish that for making an optimum decision, the careful consideration  
 13 must be given to the choice of the loss function and one should not just consider appropriate prior  
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TABLE 9. 95% prediction intervals and point predictors for  $X_{i:m:n:s}^R$  when  $k = 1$  and  $(a, b) = (0.5, 0.5)$ .

CS	$i$	Prediction interval		Point predictor		
		Lower	Upper	SE	LINEX	
				$\vartheta = 0.5$		$\vartheta = 1$
I	1	0.0249	6.41069	1.372	0.92544	0.75168
	2	0.34552	19.6229	4.802	2.54114	1.94704
	5	4.63435	107.639	29.9553	9.35748	6.64666
II	1	0.0249	6.41069	1.372	0.92544	0.75168
	2	0.22772	11.4318	2.89644	1.84024	1.47887
	5	1.58548	30.2426	8.85811	4.74453	3.70847
III	8	6.29625	117.997	34.0114	11.0019	7.97905
	1	0.01245	3.20534	0.686	0.53721	0.46272
	5	0.79274	15.1213	4.42906	2.88621	2.37226
IV	8	2.09218	32.2619	9.80272	5.36927	4.25876
	10	4.1676	67.5167	20.0927	8.50762	6.45626
	1	0.00415	1.06845	0.22867	0.20615	0.19088
V	5	0.22192	4.17791	1.22814	1.02049	0.91275
	8	0.50409	7.51108	2.30287	1.78084	1.55317
	10	0.93835	14.4759	4.36087	2.9463	2.46621
VI	1	0.00277	0.7123	0.15244	0.14165	0.13369
	5	0.16196	3.06649	0.90001	0.77625	0.70615
	8	0.39885	6.02192	1.84008	1.47205	1.29991
VII	10	0.75676	11.8414	3.55508	2.50422	2.12069
	1	0.00277	0.7123	0.15244	0.14165	0.13369
	5	0.15043	2.83637	0.83342	0.7251	0.66239
VIII	10	0.46421	6.29245	1.96764	1.57769	1.39644
	20	3.16558	46.4877	14.0635	6.96744	5.47593

TABLE 10. 95% prediction intervals and point predictors for  $X_{i:m:n:s}^R$  when  $k = 1$  and  $(a, b) = (1, 1)$ .

CS	$i$	Prediction interval		Point predictor		
		Lower	Upper	SE	LINEX	
				$\vartheta = 0.5$		$\vartheta = 1$
I	1	0.0221	5.27174	1.16	0.83062	0.68637
	2	0.30986	15.9935	4.06	2.33045	1.81293
	5	4.20602	86.7622	25.3267	8.84877	6.35123
II	1	0.0221	5.27174	1.16	0.83062	0.68637
	2	0.20428	9.28556	2.44889	1.66607	1.36148
	5	1.44197	24.2321	7.48937	4.37811	3.47413
III	8	5.73966	94.7871	28.756	10.4117	7.62936
	1	0.01105	2.63587	0.58	0.47426	0.41531
	5	0.72098	12.1161	3.74468	2.61968	2.18905
IV	8	1.91205	25.7465	8.28802	4.95706	3.99095
	10	3.80932	54.0393	16.988	7.97251	6.12585
	1	0.00368	0.87862	0.19333	0.17821	0.16705
V	5	0.20186	3.34608	1.03837	0.89844	0.81653
	8	0.46094	5.98711	1.94703	1.54238	1.40795
	10	0.85838	11.5689	3.68703	2.65471	2.26997
VI	1	0.00246	0.58575	0.12889	0.12176	0.11612
	5	0.14731	2.45643	0.76094	0.67906	0.62709
	8	0.36463	4.80221	1.55575	1.30537	1.17203
VII	10	0.69209	9.4674	3.00575	2.25652	1.9429
	1	0.00246	0.58575	0.12889	0.12176	0.11612
	5	0.13683	2.27176	0.70464	0.63333	0.58717
VIII	10	0.42542	5.00336	1.6636	1.39984	1.25991
	20	2.90145	37.1436	11.8905	6.4773	5.16029

TABLE 11. 95% prediction intervals and point predictors for  $X_{i:m:n:s}^R$  when  $k = 1$  and  $(a, b) = (2, 2)$ .

CS	$i$	Prediction interval		Point predictor		
		Lower	Upper	SE	LINEX	
				$\vartheta = 0.5$		$\vartheta = 1$
I	1	0.01817	3.90678	0.895	0.69268	0.58805
	2	0.2588	11.7032	3.1325	2.00917	1.60276
	5	3.57761	62.4403	19.5408	8.02138	5.86283
II	1	0.01817	3.90678	0.895	0.69268	0.58805
	2	0.1707	6.75879	1.88944	1.40661	1.18075
	5	1.23054	17.2714	5.77843	3.80549	3.09625
III	1	0.00909	1.95339	0.4475	0.38597	0.34634
	5	0.61527	8.63572	2.88922	2.21718	1.90274
	8	1.64457	18.233	6.39463	4.30984	3.55814
IV	1	0.00303	0.65113	0.14917	0.14098	0.13432
	5	0.1723	2.38314	0.80115	0.72687	0.67587
	8	0.39679	4.23146	1.50224	1.30613	1.18853
V	1	0.00202	0.43409	0.09944	0.09567	0.09242
	5	0.12573	1.75008	0.5871	0.54469	0.51355
	8	0.31378	3.39655	1.20035	1.06635	0.9812
VI	1	0.00202	0.43409	0.09944	0.09567	0.09242
	5	0.11679	1.61814	0.54367	0.50698	0.4796
	10	0.36758	3.52156	1.28355	1.44413	1.05551
	20	2.50791	26.3644	9.1741	5.69514	4.64417

TABLE 12. 95% prediction intervals and point predictors for  $X_{i:m:n:s}^R$  when  $k = 2$  and  $(a, b) = (0.5, 0.5)$ .

CS	$i$	Prediction interval		Point predictor		
		Lower	Upper	SE	LINEX	
				$\vartheta = 0.5$		$\vartheta = 1$
I	1	0.02782	6.26078	1.40857	0.97578	0.79333
	2	0.39337	18.8613	4.93	2.70565	2.07091
	5	5.39272	101.395	30.7538	10.1172	7.15811
II	1	0.02782	6.26078	1.40857	0.97578	0.79333
	2	0.25941	10.9192	2.97365	1.9594	1.57716
	5	1.85207	28.1731	9.09423	5.15094	4.03323
III	1	0.01391	3.13039	0.70429	0.56421	0.48789
	5	0.92603	14.0866	4.54711	3.11531	2.5755
	8	2.46617	29.8309	10.064	5.87067	4.67704
IV	1	0.00464	1.04346	0.23476	0.2146	0.19984
	5	0.2593	3.88873	1.26088	1.08251	0.97812
	8	0.59478	6.92958	2.36426	1.91033	1.68445
V	1	0.00309	0.69564	0.15651	0.14701	0.1395
	5	0.18922	2.8553	0.924	0.81966	0.7533
	8	0.47043	5.56035	1.88913	1.57309	1.40476
VI	1	0.89323	10.9877	3.64985	2.70272	2.31051
	5	0.00309	0.69564	0.15651	0.14701	0.1395
	10	0.17576	2.64032	0.85563	0.76481	0.70585
	20	0.55004	5.77832	2.02008	2.77549	1.51336
		3.75202	43.0861	14.4384	7.66772	6.06231

TABLE 13. 95% prediction intervals and point predictors for  $X_{i:m:n:s}^R$  when  $k = 2$  and  $(a, b) = (1, 1)$ .

CS	$i$	Prediction interval		Point predictor		
		Lower	Upper	SE	LINEX	
					$\vartheta = 0.5$	$\vartheta = 1$
I	1	0.02528	5.43457	1.245	0.89812	0.73976
	2	0.36001	16.2798	4.3575	2.52923	1.95856
	5	4.97668	86.8583	27.1825	9.67337	6.9003
II	1	0.02528	5.43457	1.245	0.89812	0.73976
	2	0.23746	9.40189	2.62833	1.81367	1.47859
	5	1.71175	24.0256	8.03815	4.83182	3.82752
III	8	6.83967	94.3945	30.8632	11.4956	8.39332
	1	0.01264	2.71728	0.6225	0.51302	0.44906
	5	0.85588	12.0128	4.01908	2.88576	2.41591
IV	8	2.28769	25.3633	8.89533	5.50661	4.43806
	10	4.55907	53.502	18.2328	8.86358	6.80511
	1	0.00421	0.90576	0.2075	0.19229	0.18059
V	5	0.23968	3.31509	1.11446	0.98082	0.89634
	8	0.55196	5.88621	2.08971	1.74538	1.55814
	10	1.02869	11.4255	3.95721	2.94847	2.52112
VI	1	0.00281	0.60384	0.13833	0.13124	0.12539
	5	0.17489	2.43447	0.8167	0.73943	0.6867
	8	0.43649	4.72481	1.66975	1.43207	1.29427
VII	10	0.82906	9.3565	3.226	2.48915	2.15444
	1	0.00281	0.60384	0.13833	0.13124	0.12539
	5	0.16246	2.25093	0.75627	0.68924	0.64261
VIII	10	0.51132	4.89871	1.7855	1.52902	1.39472
	20	3.48866	36.6745	12.7617	7.22771	5.77448

TABLE 14. 95% prediction intervals and point predictors for  $X_{i:m:n:s}^R$  when  $k = 2$  and  $(a, b) = (2, 2)$ .

CS	$i$	Prediction interval		Point predictor		
		Lower	Upper	SE	LINEX	
					$\vartheta = 0.5$	$\vartheta = 1$
I	1	0.02148	4.3145	1.016	0.7772	0.65407
	2	0.30917	12.8144	3.556	2.2448	1.7735
	5	4.33122	67.5645	22.1827	8.92216	6.45819
II	1	0.02148	4.3145	1.016	0.7772	0.65407
	2	0.20399	7.37182	2.14489	1.58301	1.31847
	5	1.49337	18.5493	6.55965	4.30843	3.48256
III	8	5.98523	73.1455	25.1863	10.6014	7.85424
	1	0.01074	2.15725	0.508	0.43551	0.3886
	5	0.74668	9.27464	3.27983	2.51942	2.15421
IV	8	2.00809	19.4855	7.25916	4.90774	4.03576
	10	4.00307	41.273	14.8792	8.05928	6.29483
	1	0.00358	0.71908	0.16933	0.15975	0.15188
V	5	0.20913	2.55793	0.90947	0.8271	0.76899
	8	0.48481	4.51474	1.70533	1.49756	1.36445
	10	0.90417	8.79669	3.22933	2.56578	2.24017
VI	1	0.00239	0.47939	0.11289	0.10848	0.10465
	5	0.1526	1.87894	0.66648	0.61969	0.58445
	8	0.38329	3.62615	1.36263	1.21599	1.11969
VII	10	0.72847	7.20768	2.63263	2.1524	1.90114
	1	0.00239	0.47939	0.11289	0.10848	0.10465
	5	0.14175	1.73695	0.61717	0.57677	0.54584
VIII	10	0.45046	3.74476	1.45708	1.3036	1.21045
	20	3.07476	28.2347	10.4144	6.49547	5.2373