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PARAMETER ESTIMATION METHODS FOR TWO-COMPONENT MIXED EXPONENTIAL DISTRIBUTIONS

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Abstract: In this paper, parameter estimation of mixture of exponential distributions with two-component, will be made by method of moments, maximum likelihood and least square. Comprise of these three methods will be made by simulation techniques and the root of mean square errors (RMSE). We will compare performance of methods through numerical simulations.

Key words: EM-Algorithm, Least Squares Method, Maximum Likelihood Method, Method of Moments, Mixed Exponential Distributions, Parameter Estimation
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1. Introduction

One can generally point out that data set has only one law in modelling and estimation problems. However, it's possible that obtained data appear as either mixture of same distribution family or contaminated data set. For example, survival times of individuals having same illness which includes two different risks or length of stay in hospital of patients who stay a specific unit, the time until considering specific characteristics of married couples have their first child, intensity of the daily raining (amount of precipitation per m^2), the time till the guilty commits the same crime (types of crime), divorce time of couples having different socio-cultural, first failure time of products that have same function in different qualities can be modelled with mixture of exponential distribution.

There are two disadvantages while making parameter estimation of mixture distribution. The first of these, the more considering components of mixture distribution the more increasing number of parameter estimation. In general, this disadvantage uncovers solving problems on software with algorithm of nonlinear equations. The second, the model which consists of two mixture distribution with very close shape parameters may interpret as one distribution. In this case, it may be possible to ignore this disadvantage by putting necessary and appropriate conditions on the parameters. For instance, the person buys a box, included light bulbs whose life times are 3 or 4 years with same power, but he does not know their life time and wants to determine it. In this case the person cannot determine that the life time comes from two different distributions since it's difficult to distinguish two nested distributions.

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2. Definitions

In the literature, [1] and [4] have mentioned finite mixture distributions and methods of parameter estimation for them. In this section, mixed exponential distribution with two-component will be considered.

2.1. Mixed exponential distribution with two-components

Probability density function (pdf) of mixed exponential distribution with two-component is given below.

$$f(x;\alpha,\theta_1,\theta_2)=\alpha f_1(x;\theta_1)+(1-\alpha)f_2(x;\theta_2)=\alpha\ \frac{1}{\theta_1}\ e^{-x\frac{1}{\theta_1}}\ +(1-\alpha)\ \frac{1}{\theta_2}\ e^{-x\frac{1}{\theta_2}}$$

where $\alpha\in(0,1)$, $\theta_i>0$ $(i=1,2)$, $x>0$. Similarly the cumulative distribution function (cdf) is as follows.

$$F(x;\alpha,\theta_1,\theta_2)=\alpha F_1(x;\theta_1)+(1-\alpha)F_2(x;\theta_2)=\alpha(1-e^{-x\frac{1}{\theta_1}})+(1-\alpha)(1-e^{-x\frac{1}{\theta_2}})$$

Survival function,

$$S(x;\alpha,\theta_1,\theta_2)=\alpha S_1(x;\theta_1)+(1-\alpha)S_2(x;\theta_2)=\alpha\ e^{-x\frac{1}{\theta_1}}+\ (1-\alpha)\ e^{-x\frac{1}{\theta_2}}$$

and the hazard function,

$$\begin{aligned} h(x;\alpha,\theta_1,\theta_2) &= \frac{\alpha h_1(t)S_1(x;\theta_1)+\ (1-\alpha)h_2(t)S_2(x;\theta_2)}{\alpha S_1(x;\theta_1)+(1-\alpha)S_2(x;\theta_2)} \\ &= h_1(t)\frac{\alpha S_1(x;\theta_1)}{\alpha S_1(x;\theta_1)+(1-\alpha)S_2(x;\theta_2)}+h_2(t)\ \frac{(1-\alpha)S_2(x;\theta_2)}{\alpha S_1(x;\theta_1)+(1-\alpha)S_2(x;\theta_2)} \\ &= h_1(t)w_1(x;\alpha,\theta_1,\theta_2)+h_2(t)w_2(x;\alpha,\theta_1,\theta_2) \end{aligned}$$

where $h_i(t)=\frac{1}{\theta_i}$ and $w_1+w_2=1$ ([6], [7], [8] and [9]).

3. Parameter Estimation Methods

In this section, methods of moments (MOM), maximum likelihood estimation (MLE) and least square estimation (LSE) are given for two-component mixed exponential distribution.

3.1. Methods of moments (MOM)

Since two-component mixed exponential distribution has three parameters such as α , θ_1,θ_2, the first three sample moments are equalized to the first three population moments.

$$\begin{aligned} m_1 &= \frac{\sum_{i=1}^{n}x_i}{n}=\alpha\ \ \theta_1+(1-\alpha)\theta_2 \\ m_2 &= \frac{\sum_{i=1}^{n}x_i^2}{n}=2\alpha\ \ {\theta_1}^2+2(1-\alpha)\ {\theta_2}^2 \\ m_3 &= \frac{\sum_{i=1}^{n}x_i^3}{n}=6\alpha\ \ {\theta_1}^3+6(1-\alpha)\ {\theta_2}^3 \end{aligned}$$

By arranging the equalities,

$$m_1 = \alpha\ \ \theta_1+(1-\alpha)\ \theta_2 \tag{3.1}$$
$$\frac{m_2}{2} = \alpha\ \ {\theta_1}^2+(1-\alpha)\ {\theta_2}^2 \tag{3.2}$$
$$\frac{m_3}{6} = \alpha\ \ {\theta_1}^3+(1-\alpha)\ {\theta_2}^3 \tag{3.3}$$

Then we have equations as below:

$$\begin{aligned} \frac{m_3}{6} &= \frac{m_2}{2}\ (\theta_1+\theta_2)-\theta_1\theta_2 m_1 \\ \frac{m_2}{2} &= m_1\,(\theta_1+\theta_2)-\theta_1\theta_2 \end{aligned} \tag{3.4}$$

(3.4) are solved by letting $\theta_1+\theta_2=z_1$ and $\theta_1\theta_2=z_2$, we obtain

$$z_1 = \left(\frac{m_3}{6}-\frac{m_1m_2}{2}\right)/\left(\frac{m_2}{2}-m_1^2\right) \tag{3.5}$$

$$z_2 = \left(\frac{m_1m_3}{6}-\frac{{m_2}^2}{4}\right)/\left(\frac{m_2}{2}-m_1^2\right) \tag{3.6}$$

Due to the definitions of $z_1$and z_2, obtained solutions must always be positive. Thus, as mentioned in [3], the conditions (i) and (ii) are given below.

i. If $\frac{m_2}{2}>m_1^2$, $\frac{m_3}{6}>\frac{m_1m_2}{2}$ and $\frac{m_3}{6}m_1>(\frac{m_2}{2})^2$ which implies $\frac{1}{m_1}>\frac{m_1}{\frac{m_2}{2}}>\frac{\frac{m_2}{2}}{\frac{m_3}{6}}$

ii. If $\frac{m_2}{2}<m_1^2$, $\frac{m_3}{6}<\frac{m_1m_2}{2}$ and $\frac{m_3}{6}m_1<(\frac{m_2}{2})^2$ which implies $\frac{1}{m_1}<\frac{m_1}{\frac{m_2}{2}}<\frac{\frac{m_2}{2}}{\frac{m_3}{6}}$

In order to be able to get the easier solution, another equality such as $\theta_1-\theta_2=\pm\sqrt{z_1^2-4z_2}$ is required as well as z_1 and z_2 . If $\theta_1>\theta_2$, the positive part of $\pm\sqrt{z_1^2-4z_2}$ is used for the solution. Then the estimation process will be followed straightforwardly the equalities $\theta_1+\theta_2=z_1$ and $\theta_1-\theta_2=\sqrt{z_1^2-4z_2}$.

To be noted that one more condition is needed. The condition is that $z_1^2-4z_2$ must be positive valued. Conditions (i) and (ii) given above will be checked whether they are sufficient or not. In case of sample moments there is a linear relationship between z_1 and z_2 as

$$z_2=m_1z_1+\frac{{m_1}^2\frac{m_2}{2}-(\frac{m_2}{2})^2}{\frac{m_2}{2}-{m_1}^2}$$

thus,

$$z_2=m_1z_1-\frac{m_2}{2} \tag{3.7}$$

Then

$$z_1^2-4z_2=(z_1-2m_1)^2+4(\frac{m_2}{2}-{m_1}^2) \tag{3.8}$$

and if $\frac{m_2}{2}>m_1^2$, $z_1^2-4z_2\geq 0$. In case of (ii), which is $\frac{m_2}{2}<m_1^2$, the expression $(z_1-2m_1)^2-4({m_1}^2-\frac{m_2}{2})$ is a convex function of z_1. It has min point at ${z_1}^*=2m_1$ and the value of this convex function is negative signed at this point. This means that the statement $z_1^2-4z_2$ can take negative values in some situations. Therefore, condition (ii) is not enough alone. In addition to (ii), $z_1^2\geq 4z_2$is provided for real solutions of θ_1 and θ_2. Moreover, even if condition (ii) is satisfied, this condition and $z_1^2\geq 4z_2$ can not guarantee the range of mixing parameter since ${m_1}^2-\frac{m_2}{2}=\alpha\,(1-\alpha)\,(\theta_1-\theta_2)^2$. In case of $\theta_1>\theta_2$,

$$\hat{\theta}_1=\frac{z_1+\sqrt{z_1^2-4z_2}}{2} \text{ and } \hat{\theta}_2=\frac{z_1-\sqrt{z_1^2-4z_2}}{2} \tag{3.9}$$

and, from $m_1=\alpha\hat{\theta}_1+(1-\alpha)\hat{\theta}_2$

$$\hat{\alpha}=\frac{m_1-\hat{\theta}_2}{\hat{\theta}_1-\hat{\theta}_2} \tag{3.10}$$

are obtained. An illustrative example is given below in order to support the above argument such that two simulated data sets containing 50 sample units are generated with the population parameters as $\theta_1=5$, $\theta_2=3$ and $\alpha=0.60$.

TABLE 1. Simulated data sets

Data1									
8.31	7.8	4.17	3.42	20.06	1.1	9.67	0.04	0.13	4.41
6.97	3.67	0.18	8.28	13.65	7.97	6.72	15.99	0.27	5.32
1.39	0.11	20.95	4.42	0.35	0.4	0.63	3.86	7.26	1.3
8.4	7.73	9.92	0.01	0.61	1.27	2.64	2.61	4.21	3.2
5.54	7.11	0.48	0.27	4.85	4.82	3.23	7.63	0.12	8.35
Data2									
2.97	17.44	1.19	0.91	0.18	1.35	4.39	6.89	3.55	0.34
1.29	0.12	1.18	0.71	4.52	5.24	5.98	0.61	3.25	8.02
4.57	7.17	4.6	0.99	4.38	1.45	8.04	15.94	2.93	14.54
4.21	4.29	3.28	12.66	0.56	6.49	3.53	11.44	4.9	7.38
3.62	3.62	3.75	0.14	0.76	9.4	3.12	0.97	12.3	10.58

TABLE 2. Estimation results of Data1 and Data2

	$\frac{1}{m_1}$	$\frac{m_1}{m_{\frac{2}{2}}}$	$\frac{\frac{m_2}{2}}{\frac{m_3}{6}}$	z_1	z_2	$z_1^2-4z_2$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\alpha}$
Data1	0.20	0.21	0.23	18.81	70.16	73.05	13.68	5.13	-0.01
Data2	0.21	0.23	0.26	8.76	21.47	-9.17	-	-	-

The condition (ii) seems to be provided by both the datasets. However, while we can estimate the corresponding population parameter for data1 (except α), we bring the solution that does not belong in parameter space for data2. Therefore, we need to warn readers to be faced with unwanted results even if condition (ii) is satisfied.

3.2. Maximum likelihood estimations (MLE)

In general, MLE for parameter set $\underline{\Phi}$ is obtained by solving the likelihood equation systems with equalizing first derivative of the logarithm of the likelihood function to zero. This equation systems dont have analytical solution since they are not linear equations. Because of this, the numerical methods are preferred for the solution of likelihood equations. One of them is Newton-Raphson and the other is EM algorithm. In this paper, EM algorithm that studied in [2] [5] and [11] will be mentioned.

Let $\underline{X}=\{X_1,X_2,...,X_n\}$ be a random sampling with independent and identically distributed as two-component mixed exponential distributions having a pdf $f\left(\underline{x}\ ;\ \underline{\Phi}\right)$. $\underline{\Phi}=(\alpha,\theta_1,\theta_2)$is a parameter set of this distribution and the likelihood function of $\underline{\Phi}$ is given below.

$$L(\underline{\Phi},\underline{x})=\prod_{j=1}^{n}\left[\sum_{i=1}^{2}\alpha_i\quad\frac{1}{\theta_i}\exp(-\frac{x_j}{\theta_i})\right] \tag{3.11}$$

$$\log L=\sum_{j=1}^{n}\log\left[\sum_{i=1}^{2}\alpha_i\quad\frac{1}{\theta_i}\exp(-\frac{x_j}{\theta_i}\ )\right]-\lambda\ (\sum_{i=1}^{2}\alpha_i-1)\quad,\quad\sum_{i=1}^{2}\alpha_i=1 \tag{3.12}$$

$$\frac{d\log L}{d\alpha_i}=\sum_{j=1}^{n}\frac{\frac{1}{\theta_i}\exp(-\frac{x_j}{\theta}{}_i)}{\sum_{i=1}^{2}\alpha_i\quad\frac{1}{\theta_i}\exp(-x_j/\theta_i)}-\lambda=0$$

then

$$\sum_{j=1}^{n} \frac{\frac{1}{\theta_i} \exp(-\frac{x_j}{\theta_i})}{\sum_{i=1}^{2} \alpha_i \ \frac{1}{\theta_i} \exp(-x_j/\theta_i)} = \lambda \tag{3.13}$$

If the both side of 3.13 is multiplied with α_i and sum over index i:

$$\sum_{j=1}^{n} \underbrace{\sum_{i=1}^{2} \frac{\alpha_i \ \frac{1}{\theta_i} \exp\left(-\frac{x_j}{\theta_i}\right)}{\sum_{i=1}^{2} \alpha_i \ \frac{1}{\theta_i} \exp\left(-\frac{x_j}{\theta_i}\right)}}_{1} = \lambda \alpha_i$$

then $n = \ \lambda$. Based on Bayes rule, the probability that x_j belongs to i^{th} component when $X_j = x_j$ is observed is as follows:

$$P(i|x_j) = \sum_{j=1}^{n} \frac{\alpha_i \frac{1}{\theta_i} \exp(-x_j/\theta_i)}{\sum_{i=1}^{2} \alpha_i \ \ \frac{1}{\theta_i} \exp(-x_j/\theta_i)}$$

Thus,

$$\hat{\alpha}_i = \frac{\sum_{j=1}^{n} P(i|x_j)}{n} \ , \ i = (1, 2) \tag{3.14}$$

If the derivative of $\log L$ with respect to θ_i is equalized to zero,

$$\frac{d \log L}{d\theta_i} = \sum_{j=1}^{n} \frac{\frac{\alpha_i}{\theta_i}\left(\frac{x_j}{\theta_i{}^2}\right) \exp\left(-\frac{x_j}{\theta_i}\right) - \frac{\alpha_i}{\theta_i{}^2} \exp(-x_j/\theta_i)}{\sum_{i=1}^{2} \alpha_i \ \ \frac{1}{\theta_i} \exp(-x_j/\theta_i)} = 0$$

$$\vdots$$

$$\hat{\theta}_i = \frac{\sum_{j=1}^{n} x_j \ P(i|x_j)}{\sum_{j=1}^{n} P(i|x_j)} \ , \ i = (1, 2)$$

is obtained. Its reminded that $P(2|x_j) = 1 - P(1|x_j)$, then the solutions will be

$$\hat{\theta}_1 = \frac{1}{n\hat{\alpha}_i} \sum_{j=1}^{n} x_j \ P(i|x_j) \tag{3.15}$$

$$\hat{\theta}_2 = \frac{1}{n(1-\hat{\alpha}_i)} \sum_{j=1}^{n} x_j \ \left(1 - P(i|x_j)\right) \tag{3.16}$$

These are step solutions obtained by Expectation-Maximization algorithm (EM) which steps are given in below.

1. Input the initial values. ($\alpha_i^{(0)}$, $\theta_i^{(0)}$), $(i = 1, 2)$
2. Calculate the $P(i|x_j)$.
3. Calculate $\hat{\alpha}_i^{(k)}$, $\hat{\theta}_i^{(k)}$.
4. After calculations of $\hat{\alpha}_i$ and $\hat{\theta}_i$, the values replace in $\log L$ and get the value of function. For $\epsilon > 0$ selected small enough

$$\log L^{(k)} - \log L^{(k-1)} \leq \ \epsilon$$

is provided then the values on the k^{th} step will be used for parameter estimations. Steps 2-5 are repeated until converge is accomplished.

3.3. Least squares estimations (LSE)

This method is based on the idea that there is a regression relationship between empirical $\hat{F}$ and parametric F distributions. Considering ordered observations $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ versus empirical distribution $\hat{F}\left(x_{(i)}\right) \equiv \frac{i}{n+1}$, the vector $\underline{\Phi}$ which minimizes the following expression is tried to determine. Detailed study was given in [10] for non-mixture Generalized Exponential Distribution.

$$Q\left(\underline{\Phi}\right) = \sum_{i=1}^{n} \left(F\left(x_{(i)};\underline{\Phi}\right) - \hat{F}\left(x_{(i)}\right)\right)^2$$

$$\frac{dQ}{d\alpha} = \sum_{i=1}^{n} \left(\frac{i}{n+1} - \alpha\left(1 - e^{-x_{(i)}\frac{1}{\theta_1}}\right) - (1-\alpha)\left(1 - e^{-x_{(i)}\frac{1}{\theta_2}}\right)\right)\left(e^{-x_{(i)}\frac{1}{\theta_1}} - e^{-x_{(i)}\frac{1}{\theta_2}}\right) = 0$$

$$\frac{dQ}{d\theta_1} = \sum_{i=1}^{n} \left(\frac{i}{n+1} - \alpha\left(1 - e^{-x_{(i)}\frac{1}{\theta_1}}\right) - (1-\alpha)\left(1 - e^{-x_{(i)}\frac{1}{\theta_2}}\right)\right)\left(\frac{\alpha x_{(i)}}{\theta_1^2} e^{-x_{(i)}\frac{1}{\theta_1}}\right) = 0$$

$$\frac{dQ}{d\theta_2} = \sum_{i=1}^{n} \left(\frac{i}{n+1} - \alpha\left(1 - e^{-x_{(i)}\frac{1}{\theta_1}}\right) - (1-\alpha)\left(1 - e^{-x_{(i)}\frac{1}{\theta_2}}\right)\right)\left(\frac{(1-\alpha) x_{(i)}}{\theta_2^2} e^{-x_{(i)}\frac{1}{\theta_2}}\right) = 0$$

Since the equations that obtained after derivation are related to θ, it is difficult to obtain the solutions. Therefore it is necessary to use numerical ways.

4. Results and Discussion

Parameter estimations and RMSE values (in parenthesis) obtained by MOM, LSE and MLE are given in the following tables, respectively $n = 20,\ 40$ and 100 for repeated simulation with $r = 1000$. MATLAB is used in simulation study.

$$RMSE_\alpha = \sqrt{\frac{\sum_{k=1}^{r}\left(\hat{\alpha}_k - \alpha\right)^2}{r}} \text{ and } RMSE_{\theta_i} = \sqrt{\frac{\sum_{k=1}^{r}\left(\hat{\theta}_k - \theta_i\right)^2}{r}} \quad (i = 1, 2)$$

TABLE 3. Simulation results for $n = 20$

n	α	θ_1, θ_2	Parameter (α)			Parameter (θ_1)			Parameter (θ_2)		
			MOM	LSE	MLE	MOM	LSE	MLE	MOM	LSE	MLE
20	0.4	(10,1)	0.4625 (0.1426)	0.3647 (0.1872)	0.3003 (0.2008)	8.4145 (4.6951)	9.2826 (1.1427)	11.4051 (6.8530)	0.4954 (0.6383)	1.1427 (0.5926)	1.2969 (0.7033)
		(3,2)	0.5401 (0.2698)	0.4325 (0.2023)	0.2651 (0.2057)	4.3930 (5.8288)	2.1643 (2.2265)	5.6647 (4.0266)	0.8923 (1.2924)	2.5654 (1.0045)	1.5223 (0.7017)
		(0.8,0.3)	0.5447 (0.2425)	0.3309 (0.2193)	0.2578 (0.2130)	0.8555 (0.3478)	1.0895 (0.6238)	1.3284 (0.8833)	0.1627 (0.1864)	0.3008 (1321)	0.3070 (0.1148)
		(2,0.3)	0.4828 (0.1739)	0.3813 (0.1919)	0.2806 (0.2194)	1.8345 (0.8435)	2.1906 (1.1948)	2.7446 (1.7256)	0.1585 (0.1907)	0.3228 (0.1408)	0.3768 (0.1943)
	0.45	(10,1)	0.4805 (0.1337)	0.4198 (0.1945)	0.3501 (0.2133)	9.1633 (4.1289)	9.7134 (4.7509)	11.5160 (6.1693)	0.5539 (0.6487)	1.1657 (0.6979)	1.3455 (0.8206)
		(3,2)	0.5154 (0.2519)	0.4663 (0.2165)	0.2509 (0.2394)	4.1841 (3.1600)	2.2936 (2.2260)	5.5906 (3.7153)	1.0145 (1.2397)	2.8228 (1.3209)	1.6019 (0.6488)
		(0.8,0.3)	0.5579 (0.2219)	0.3597 (0.2375)	0.2794 (0.2442)	0.8840 (0.4225)	1.1000 (0.6347)	1.3055 (0.8269)	0.1742 (0.1862)	0.3088 (0.1361)	0.3136 (0.1291)
		(2,0.3)	0.5141 (0.1656)	0.4092 (0.2097)	0.3130 (0.2311)	1.8588 (0.8432)	2.1969 (1.1056)	2.6617 (1.6853)	0.1504 (0.1893)	0.3151 (0.1426)	0.3809 (0.2064)
	0.55	(10,1)	0.5125 (0.1400)	0.4984 (0.2030)	0.4213 (0.2438)	10.8430 (4.7665)	10.4300 (4.4257)	12.6346 (7.7415)	0.7921 (0.7677)	1.2564 (0.9401)	1.5240 (1.1800)
		(3,2)	0.5493 (0.2352)	0.5274 (0.2097)	0.2928 (0.3113)	4.2534 (2.8430)	2.4867 (2.2190)	5.7381 (3.9369)	1.0350 (1.2343)	2.9481 (1.3930)	1.6447 (0.7050)
		(0.8,0.3)	0.5535 (0.2066)	0.3807 (0.2894)	0.2905 (0.3178)	0.9730 (0.8963)	1.0841 (0.6281)	1.3628 (0.9096)	0.1779 (0.1812)	0.3206 (0.1432)	0.3258 (0.1331)
		(2,0.3)	0.5393 (0.1437)	0.4847 (0.2028)	0.3661 (0.2724)	2.0816 (0.9202)	2.2707 (1.0618)	2.7509 (1.7501)	0.1823 (0.1949)	0.3225 (0.1447)	0.4189 (0.2357)
	0.6	(10,1)	0.5086 (0.1617)	0.5491 (0.1990)	0.4382 (0.2749)	11.4268 (4.6431)	10.1646 (4.3182)	13.2679 (7.6480)	0.9186 (0.8052)	1.2650 (0.9742)	1.6227 (1.3183)
		(3,2)	0.5546 (0.2420)	0.5602 (0.2292)	0.2814 (0.3569)	4.4386 (4.7308)	2.3480 (2.1323)	5.8883 (4.0774)	1.0419 (1.2171)	3.0442 (1.4537)	1.6909 (0.6355)
		(0.8,0.3)	0.5721 (0.2141)	0.4628 (0.2802)	0.3003 (0.3511)	1.0116 (0.7135)	1.0589 (0.5701)	1.4331 (0.9933)	0.1896 (0.1946)	0.3181 (0.1555)	0.3581 (0.1571)
		(2,0.3)	0.5302 (0.1660)	0.5620 (0.2017)	0.3770 (0.3136)	2.2633 (0.9923)	2.1002 (0.9257)	2.9244 (1.8077)	0.2380 (0.1965)	0.3223 (0.1605)	0.4643 (0.3127)
	0.8	(10,1)	0.5665 (0.2929)	0.7203 (0.2350)	0.5421 (0.3815)	13.5847 (9.5814)	10.7144 (3.6990)	4.5558 (8.6339)	1.9439 (1.9996)	1.4611 (13164)	2.2906 (2.4606)
		(3,2)	0.5333 (0.3613)	0.7607 (0.3013)	0.3217 (0.5289)	4.6819 (4.1073)	2.3791 (1.8516)	6.0489 (4.2563)	1.1302 (1.1925)	2.7658 (1.4160)	1.7129 (0.7607)
		(0.8,0.3)	0.5653 (0.3265)	0.5772 (0.3752)	0.3215 (0.5238)	1.2354 (1.4786)	1.0467 (0.5469)	1.5699 (1.1134)	0.2413 (0.2025)	0.3701 (0.1690)	0.4088 (0.1967)
		(2,0.3)	0.5708 (0.2930)	0.7702 (0.1915)	0.4546 (0.4464)	2.8412 (4.2704)	2.0134 (0.6805)	3.3869 (2.4684)	0.4438 (0.3942)	0.3260 (0.1842)	0.6487 (0.5623)

TABLE 4. Simulation results for $n = 40$

n	α	θ_1, θ_2	Parameter (α)			Parameter (θ_1)			Parameter (θ_2)		
			MOM	LSE	MLE	MOM	LSE	MLE	MOM	LSE	MLE
40	0.4	(10,1)	0.4209 (0.1247)	0.3851 (0.1463)	0.3576 (0.1393)	9.8812 (4.0102)	10.5911 (4.2156)	10.6526 (4.3323)	0.6770 (0.6136)	1.0347 (0.4179)	1.1048 (0.4269)
		(3,2)	0.4700 (0.2928)	0.4303 (0.2114)	0.2693 (0.2254)	4.6190 (4.5694)	2.0270 (1.8010)	5.3505 (3.5081)	1.2812 (1.0284)	2.8956 (1.2125)	1.6942 (0.6026)
		(0.8,0.3)	0.4952 (0.2586)	0.3133 (0.2081)	0.2850 (0.2182)	0.9739 (1.6556)	1.0367 (0.6197)	1.2060 (0.7415)	0.2015 (0.1606)	0.3177 (0.1179)	0.3079 (0.1060)
		(2,0.3)	0.4818 (0.1635)	0.3798 (0.1620)	0.3496 (0.1622)	1.8767 (0.7605)	2.3354 (1.0374)	2.2609 (0.9677)	0.1792 (0.1757)	0.3150 (0.1167)	0.3368 (0.1347)
	0.45	(10,1)	0.4573 (0.1206)	0.4467 (0.1441)	0.4130 (0.1435)	10.1397 (3.4468)	10.4304 (3.8550)	10.6740 (4.0548)	0.7441 (0.6220)	1.0313 (0.4679)	1.1358 (0.4836)
		(3,2)	0.4900 (0.2907)	0.4885 (0.2180)	0.2870 (0.2558)	4.4075 (3.3746)	1.9637 (1.7483)	5.2814 (3.4724)	1.2414 (1.0560)	3.1104 (1.4070)	1.6909 (0.6180)
		(0.8,0.3)	0.5174 (0.2430)	0.3611 (0.2197)	0.3086 (0.2424)	0.9510 (1.0251)	0.9926 (0.5763)	1.1941 (0.7109)	0.2054 (0.1656)	0.3211 (0.1289)	0.3144 (0.1160)
		(2,0.3)	0.4795 (0.1527)	0.4262 (0.1728)	0.3738 (0.1850)	2.0650 (0.8259)	2.2980 (1.0004)	2.4165 (1.2273)	0.2100 (0.1778)	0.3158 (0.1243)	0.3554 (0.1669)
	0.55	(10,1)	0.4863 (0.1502)	0.5387 (0.1456)	0.4851 (0.1789)	11.2129 (3.9174)	10.0337 (3.3989)	11.1192 (4.0967)	1.0537 (0.7872)	1.0820 (0.6652)	1.2386 (0.7405)
		(3,2)	0.4976 (0.2807)	0.5328 (0.2273)	0.2967 (0.3289)	4.1961 (2.2543)	1.9796 (1.6510)	5.5654 (3.6826)	1.2492 (1.0613)	3.3292 (1.6620)	1.7088 (0.6197)
		(0.8,0.3)	0.5209 (0.2386)	0.4067 (0.2615)	0.3247 (0.3058)	1.0052 (1.1430)	1.0145 (0.5595)	1.2960 (0.8441)	0.2240 (0.1639)	0.3326 (0.1382)	0.3352 (0.1270)
		(2,0.3)	0.5107 (0.1666)	0.5291 (0.1588)	0.4321 (0.2286)	2.2757 (0.8701)	2.1739 (0.8088)	2.5727 (1.3259)	0.2708 (0.1949)	0.3232 (0.1315)	0.4036 (0.2358)
	0.6	(10,1)	0.5136 (0.1760)	0.5782 (0.1482)	0.5260 (0.1906)	11.5982 (4.6758)	10.1415 (3.2686)	11.2830 (4.5995)	1.2071 (0.9461)	1.1155 (0.7196)	1.3164 (0.9412)
		(3,2)	0.4828 (0.3064)	0.5621 (0.2537)	0.3114 (0.3622)	4.6543 (4.3766)	2.1445 (1.6745)	5.4112 (3.3611)	1.3545 (1.0122)	3.4537 (1.7732)	1.7569 (0.6476)
		(0.8,0.3)	0.5353 (0.2436)	0.4138 (0.3018)	0.3347 (0.3407)	0.9928 (0.5692)	1.0378 (0.5713)	1.3013 (0.8330)	0.2299 (0.1673)	0.3461 (0.1496)	0.3459 (0.1410)
		(2,0.3)	0.5416 (0.1723)	0.5740 (0.1581)	0.4660 (0.2421)	2.2438 (0.8041)	2.1022 (0.6974)	2.5026 (1.1846)	0.2924 (0.2210)	0.3227 (0.1400)	0.4222 (0.2715)
	0.8	(10,1)	0.5672 (0.3003)	0.7403 (0.1718)	0.6357 (0.2873)	13.2325 (5.3936)	10.6499 (3.0394)	12.9491 (6.7095)	2.4717 (2.4027)	1.6332 (1.4367)	1.9853 (2.0799)
		(3,2)	0.5019 (0.4136)	0.6815 (0.3387)	0.3352 (0.5249)	4.9972 (5.5706)	2.3517 (1.6363)	5.7884 (3.8574)	1.4097 (1.0538)	4.3959 (2.5369)	1.8413 (0.6725)
		(0.8,0.3)	0.5356 (0.3721)	0.6248 (0.3649)	0.4092 (0.4788)	1.2209 (1.5974)	1.0463 (0.5144)	1.3741 (0.8830)	0.3091 (0.2051)	0.3521 (0.1806)	0.3886 (0.2079)
		(2,0.3)	0.5722 (0.3135)	0.7879 (0.1634)	0.5481 (0.3681)	2.7554 (3.0496)	1.9914 (0.5278)	2.7426 (1.4304)	0.5524 (0.5064)	0.3151 (0.1746)	0.5947 (0.5166)

TABLE 5. Simulation results for $n = 100$

n	α	θ_1, θ_2	Parameter (α)			Parameter (θ_1)			Parameter (θ_2)		
			MOM	LSE	MLE	MOM	LSE	MLE	MOM	LSE	MLE
100	0.4	(10,1)	0.4161 (0.1202)	0.3950 (0.0990)	0.3878 (0.0863)	10.2227 (3.2827)	10.3790 (2.8106)	10.1511 (2.1438)	0.7951 (0.6147)	1.0186 (0.2716)	1.0392 (0.2399)
		(3,2)	0.4426 (0.3216)	0.1311 (0.3192)	0.3016 (0.2353)	5.1873 (8.1030)	6.6436 (3.8600)	4.8456 (2.9139)	1.4715 (0.8937)	2.0268 (0.4634)	1.7849 (0.5149)
		(0.8,0.3)	0.5063 (0.2610)	0.3164 (0.2221)	0.3472 (0.2132)	0.8530 (0.5232)	0.7893 (0.5495)	1.0330 (0.5275)	0.2185 (0.1435)	0.3448 (0.1236)	0.2959 (0.0958)
		(2,0.3)	0.4525 (0.1515)	0.3863 (0.1272)	0.3806 (0.1107)	1.9895 (0.6476)	2.2465 (0.7779)	2.1081 (0.5309)	0.2149 (0.1648)	0.3071 (0.0806)	0.3127 (0.0806)
	0.45	(10,1)	0.4444 (0.1245)	0.4580 (0.1009)	0.4378 (0.0908)	10.5008 (3.1014)	9.9054 (2.4765)	10.1262 (2.0331)	0.9045 (0.6686)	0.9864 (0.3167)	1.0531 (0.2794)
		(3,2)	0.4597 (0.3246)	0.1446 (0.3630)	0.3216 (0.2638)	5.6133 (8.9101)	6.5766 (3.8193)	5.0017 (3.3464)	1.4602 (0.9092)	2.0228 (0.5014)	1.7660 (0.5551)
		(0.8,0.3)	0.5335 (0.2550)	0.3254 (0.2518)	0.3677 (0.2245)	0.8437 (0.3131)	0.8076 (0.5592)	1.0335 (0.5114)	0.2203 (0.1471)	0.3608 (0.1304)	0.3074 (0.0972)
		(2,0.3)	0.4813 (0.1433)	0.4473 (0.1297)	0.4283 (0.1168)	2.0221 (0.5954)	2.1395 (0.6759)	2.0974 (0.5075)	0.2320 (0.1737)	0.3017 (0.0933)	0.3177 (0.0968)
	0.55	(10,1)	0.4915 (0.1440)	0.5422 (0.0945)	0.5264 (0.0959)	11.1286 (3.1865)	10.0351 (2.2258)	10.2925 (1.9281)	1.1523 (0.8599)	1.0075 (0.3693)	1.0611 (0.3499)
		(3,2)	0.4609 (0.3292)	0.1428 (0.4551)	0.3593 (0.3098)	4.9482 (5.8862)	6.6834 (3.9064)	4.9308 (3.2019)	1.5128 (0.8885)	2.0826 (0.5470)	1.7860 (0.5937)
		(0.8,0.3)	0.5307 (0.2494)	0.4147 (0.2649)	0.3964 (0.2675)	0.9685 (1.6934)	0.8961 (0.5068)	1.0793 (0.5486)	0.2479 (0.1498)	0.3552 (0.1329)	0.3267 (0.1176)
		(2,0.3)	0.5189 (0.1535)	0.5300 (0.1221)	0.5054 (0.1347)	2.1596 (0.6283)	2.0981 (0.5375)	2.1376 (0.5283)	0.2900 (0.2030)	0.3143 (0.1051)	0.3376 (0.1296)
	0.6	(10,1)	0.5259 (0.1618)	0.5950 (0.1002)	0.5762 (0.1056)	11.2773 (3.1413)	10.0218 (2.1005)	10.3418 (1.9035)	1.3062 (1.0354)	1.0390 (0.4103)	1.1177 (0.4446)
		(3,2)	0.4693 (0.3477)	0.1608 (0.4930)	0.3710 (0.3505)	4.7091 (4.0436)	6.6691 (3.9343)	4.9936 (3.2977)	1.5205 (0.9224)	2.0962 (0.6033)	1.7398 (0.6726)
		(0.8,0.3)	0.5428 (0.2531)	0.4798 (0.2693)	0.4307 (0.2922)	0.9876 (0.8397)	0.9418 (0.4940)	1.0937 (0.5536)	0.2634 (0.1582)	0.3445 (0.1376)	0.3314 (0.1356)
		(2,0.3)	0.5477 (0.1610)	05804 (0.1249)	0.5500 (0.1460)	2.2033 (0.6083)	2.0775 (0.4538)	2.1517 (0.4949)	0.3179 (0.2172)	0.3172 (0.1092)	0.3482 (0.1527)
	0.8	(10,1)	0.6083 (0.2669)	0.7678 (0.1198)	0.7353 (0.1673)	12.1512 (3.6631)	10.2851 (1.7799)	10.8005 (2.3219)	2.5468 (2.3293)	1.2415 (0.9560)	1.3926 (1.2081)
		(3,2)	0.4689 (0.4647)	0.1886 (0.6708)	0.4674 (0.4531)	5.5492 (7.8306)	6.6697 (3.9898)	4.9904 (3.1741)	1.6638 (0.9073)	2.1943 (0.8081)	1.7202 (0.8026)
		(0.8,0.3)	0.5063 (0.4148)	0.6668 (0.3396)	0.4882 (0.4279)	1.2956 (3.7787)	1.0138 (0.4934)	1.2197 (0.7392)	0.3611 (0.2151)	0.3518 (0.1840)	0.3708 (0.1986)
		(2,0.3)	0.5852 (0.3076)	0.7919 (0.1117)	0.6709 (0.2519)	2.5030 (0.9819)	1.9868 (0.3277)	2.3514 (0.8764)	0.5974 (0.4930)	0.3062 (0.1544)	0.4543 (0.3688)

In addition to simulation studies, these methods are applied on a real data. Following table below includes the time intervals (in days) of the successive earthquakes with magnitudes greater than or equal to 6Mw.

TABLE 6. Time intervals of the successive earthquakes in North Anatolia fault zone

1163	3258	323	159	756	409
501	616	398	67	896	8592
2039	217	9	633	461	1821
4863	143	182	2117	3709	979

(Source: [12])

Parameter estimations, KS statistics and p values obtained by MOM, LSE and MLE are given in the table below.

TABLE 7. Parameter estimations for raw data

MOM			LSE			MLE		
$\hat{\alpha}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\alpha}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\alpha}$	$\hat{\theta}_1$	$\hat{\theta}_2$
0.7	2068.0	85.6	0.3	4058.5	592.9	0.4519	2549.37	506.134
KS Stat.		**p value**	**KS Stat.**		**p value**	**KS Stat.**		**p value**
0.2234		0.1559	0.0729		0.9985	0.0762		0.9971

As seen in Table 7 LSE gives the best estimates according to Kolmogrov-Simirnov statistics.

5. Conclusions

In simulation studies by considering MOM, the condition (*ii*), proposed in [3], is not sufficient alone. Thus this situation is discussed by suggesting another condition. This additional condition is important in terms of the usability of parameter estimators obtained by MOM.

Examining the estimators for small samples in results of simulation study, MOM and LSE give better estimations with regard to RMSE when the averages of the distribution diverge from each other. In contrast, LSE is found the most powerful method but MLE is better to estimate θ_2.

Examining the estimators for large samples, MLE is found the best method with regard to RMSE. Considering the other two methods, LSE gives better estimation than MOM.

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