

# IMPROVED ESTIMATORS OF THE EXPONENTIAL MODEL BASED ON TYPE TWO CENSORED DATA

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**Abstract:** This study presents two different kinds of preliminary test estimators based on Type II censored observations in the two parameters exponential model. We define MLE and MRE preliminary test estimators in the same fashion as in the ordinary preliminary test estimator using relevant combinations of MLE and MRE estimators. Exact bias and MSE expressions for the proposed estimators are derived. We compare the MSEs and obtain some intervals for the parameter of interest in which the preliminary test type estimators outperforms the MLE and MRE estimators. Some graphical representations are given for the illustration purpose. Finally, we conclude this approach by a useful discussion for practical purposes and a summary.

**Key words:** MLE and MRE preliminary test estimators, Type II censored observations, exponential model, Relative Efficiency.

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## 1. Introduction

The two parameters exponential distribution has many applications in real world situations. It can be used to model the data such as service time of agents in a system (Queuing Theory), the time it takes before your next telephone call, the time until a radioactive particle decays, the distance between mutations on a DNA strand, and the extreme values of annual snowfall or rainfall. The probability density function (p.d.f) and cumulative distribution function of the two parameters exponential distribution are respectively given by

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, \quad x > \mu, \sigma > 0, \mu \in R. \quad (1.1)$$

Due to its importance many authors have considered the estimation of the parameters of this distribution. Lawless (1982) and Mann et al. (1974) considered the earlier studies upon the estimation of exponential parameters. More recent works can also be found in Johnson et al. (1982) and Balakrishnan and Basu (1995).

Censoring is an effective way of gathering the data when the we concern with some instrument or time. One of the most important censoring scheme is Type II censoring in which the experiment will ended as soon some pre-specified failures. Suppose  $n$  item are put in the life test and once the first  $r$  out of  $n$  occurs the experimenter stop the test. Then the joint distribution of the first  $r$  observations form a p.d.f  $f_X(x; \theta, \mu)$  and c.d.f  $F_X(x; \theta, \mu)$  is given by

$$f_{X(1), X(2), \dots, X(r)}(\mathbf{x}) = \frac{n!}{(n-r)!} \prod_{i=1}^r f(x_i; \theta, \mu) (1 - F(x_{(r)}; \theta, \mu))^{n-r}, \quad (1.2)$$

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for  $X_{(1)} < X_{(2)} < \dots < X_{(r)}$ .

In the problems of statistical inference, there may exist some known prior information on some (all) of the parameters, which are usually incorporated in the model as a constraint, giving rise to restricted models. The estimators resulting from restricted (unrestricted) model is known as the restricted (unrestricted) estimators. Mostly the validity of a restricted estimator is under suspicion, resulting to make a preliminary test on the restrictions. Bancroft (1944) pioneered the use of the preliminary test estimator (PTE) to eliminate such doubt, and further developments appeared in the works of Saleh and Sen (1978), Saleh and Kibria (1993), Kibria (2004), Kibria and Saleh (1993, 2004, 2005, 2006, 2010) and Saleh (2006).

## 2. Preliminary Test Estimators

In this section we define two different types of PTEs for the scale parameter ( $\theta$ ) of exponential model distribution assuming  $\mu$  to be known. Let  $X_{(1)}, \dots, X_{(r)}$  be the first  $r$  ordered censored observations from the exponential model. Then the MLE of  $\theta$  denoted  $\hat{\theta}_{ML}$  is as follows (Balakrishnan (1990)),

$$\hat{\theta}_{ML} = \frac{1}{r} \left[ \sum_{i=1}^r (X_{(i)} - \mu) + (n - r) (X_{(r)} - \mu) \right]. \quad (2.1)$$

Since  $\frac{2r\hat{\theta}_{ML}}{\theta}$ , distributed as the chi-square random variable with  $2r$  degrees of freedom therefore the best invariant estimator of  $\theta$  denoted by  $\hat{\theta}_{MR}$  is as follows:

$$\hat{\theta}_{MR} = \frac{r}{r+1} \hat{\theta}_{ML}. \quad (2.2)$$

In the sequel we define PTEs based on MLE, and MRE of  $\theta$  when it is suspected that  $\theta$  may be equal to  $\theta_0$ . Often the information on the value of  $\theta$  is available from the past knowledge or experiments. This non-sample prior information can be expressed in the form of the following group of hypotheses:

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0 \end{cases}.$$

It is common to name the estimator  $\hat{\theta} = \theta_0$  as the restricted estimator (RE) of  $\theta$ . Then based on classical hypothesis testing, we will reject  $H_0$  if

$$\frac{2r\hat{\theta}_{ML}}{\theta_0} < c_1 = \chi_{2r}^2(\alpha/2) \quad \text{or} \quad \frac{2r\hat{\theta}_{ML}}{\theta_0} > c_2 = \chi_{2r}^2(1 - \alpha/2) \quad (2.3)$$

where  $\alpha$  is the level of significance and  $c_1, c_2$  are the critical values of central chi-square distribution with  $2r$  degrees of freedom. It should be mentioned here that Baklizi (2005) obtained the PTE of the scale parameter using the minimum regret criteria. But he did not consider which kind of PTE perform better. Also the PTE for the location parameter neglected. Further Arabi Belaghi et al. (2015) developed the preliminary test estimators for the Burr Type XII model based on the upper records. Thus by making use of the method in Arabi Belaghi et al. (2015), we define two different PTEs for  $\theta$  based on MLE and MRE namely, MLPTE and MRPTE respectively as:

$$\hat{\theta}_{ML}^{PT} = \hat{\theta}_{ML} - \left( \hat{\theta}_{ML} - \theta_0 \right) I(A), \quad (2.4)$$

$$\hat{\theta}_{MR}^{PT} = \hat{\theta}_{MR} - \left( \hat{\theta}_{MR} - \theta_0 \right) I(A), \quad (2.5)$$

where  $I(A)$  is the indicator function of the set

$$A = \{ \chi_{2r}^2 : c_1 \leq \chi_{2r}^2 \leq c_2 \}, \quad c_1 = \chi_{2r}^2(\alpha/2), \quad c_2 = \chi_{2r}^2(1 - \alpha/2).$$

### 3. Bias and MSE Expressions

In this section we obtain the bias and MSE functions for MLPTE, MRPTE.

Assuming  $\lambda = \frac{\theta_0}{\theta}$  we have

$$\begin{aligned} \text{Bias} \left( \hat{\theta}_{ML}^{PT} \right) &= E \left( \hat{\theta}_{ML} - \left( \hat{\theta}_{ML} - \theta \right) I(A) - \theta \right) \\ &= \theta_0 (H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda)) - \theta (H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)), \end{aligned}$$

where  $H_\nu(\cdot)$  stands for the cdf of  $\chi^2$  distribution with  $\nu$  degrees of freedom.

Also  $\text{MSE} \left( \hat{\theta}_{ML}^{PT} \right) = E \left( \hat{\theta}_{ML}^{PT} - \theta \right)^2 = \text{Var} \left( \hat{\theta}_{ML}^{PT} \right) + \left( \text{Bias} \left( \hat{\theta}_{ML}^{PT} \right) \right)^2$ . Since

$$\begin{aligned} \text{Var} \left( \hat{\theta}_{ML}^{PT} \right) &= \text{Var} \left( \hat{\theta}_{ML} \right) + \text{Var} \left( \left( \hat{\theta}_{ML} - \theta_0 \right) I(A) \right) - 2\text{Cov} \left( \hat{\theta}_{ML}, \left( \hat{\theta}_{ML} - \theta_0 \right) I(A) \right) \\ &= \frac{\theta^2}{r} + \frac{r+1}{r} \theta^2 [H_{2r+4}(c_2\lambda) - H_{2r+4}(c_1\lambda)] - \theta^2 [H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)]^2 \\ &\quad + \theta_0^2 \left\{ \left( H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda) \right) \left( 1 - H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda) \right) \right. \\ &\quad \left. - 2\theta_0 \theta \left( H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda) \right) \left[ H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda) \right] \right. \\ &\quad \left. - 2\theta^2 \left\{ \frac{r+1}{r} \left[ H_{2r+4}(c_2\lambda) - H_{2r+4}(c_1\lambda) \right] - \left[ H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda) \right] \right\} \right. \\ &\quad \left. + 2\theta_0 \theta \left[ H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda) \right] \right\} \end{aligned}$$

the MSE can be simplified to

$$\begin{aligned} \text{MSE} \left( \hat{\theta}_{ML}^{PT} \right) &= \frac{\theta^2}{r} - \frac{r+1}{r} \theta^2 [H_{2r+4}(c_2\lambda) - H_{2r+4}(c_1\lambda)] \\ &\quad + 2\theta^2 [H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)] \\ &\quad + \theta^2 (\lambda^2 - 2\lambda) (H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda)). \end{aligned} \tag{3.1}$$

Also the Bias and MSE of the MRPTE is given by

$$\begin{aligned} \text{Bias} \left( \hat{\theta}_{MR}^{PT} \right) &= -\theta \left( \frac{1}{r+1} + \frac{r}{r+1} (H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)) \right), \\ \text{MSE} \left( \hat{\theta}_{MR}^{PT} \right) &= \frac{\theta^2}{(r+1)^2} - \frac{r}{r+1} \theta^2 (H_{2r+4}(c_2\lambda) - H_{2r+4}(c_1\lambda)) \\ &\quad + \theta_0^2 (H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda))^2 \\ &\quad + 2 \left( \frac{r}{r+1} \right) \theta^2 (H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)) \\ &\quad + 2 \frac{r}{(r+1)^2} \theta^2 (H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)) \\ &\quad - 2\theta_0 \theta (H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda)) + \frac{\theta^2}{(r+1)^2}. \end{aligned}$$

**4. Comparison of The Estimators** In this part we compare two different sorts of proposed estimators based on the relative efficiencies. The relative efficiency (r.e) of  $\hat{\theta}_{ML}^{PT}$  with respect to  $\hat{\theta}_{ML}$  is

$$\begin{aligned} r.e \left( \hat{\theta}_{ML}^{PT}, \hat{\theta}_{ML} \right) &= \frac{\text{MSE} \left( \hat{\theta}_{ML} \right)}{\text{MSE} \left( \hat{\theta}_{ML}^{PT} \right)} \\ &= \left\{ 1 - (r+1) [H_{2r+4}(c_2\lambda) - H_{2r+4}(c_1\lambda)] \right. \\ &\quad \left. + 2r [H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)] \right. \\ &\quad \left. + r (H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda)) (\lambda^2 - 2\lambda) \right\}^{-1}. \end{aligned}$$

TABLE 1.  $\hat{\theta}_{ML}^{PT}$  dominate  $\hat{\theta}_{ML}$  for different sample size and  $\alpha$  level

r	$\alpha$						
	0.01	0.05	0.10	0.15	0.20	0.25	0.50
2	[0.507,1.703]	[0.539, 1.683]	[0.539, 1.657]	[0.526, 1.628]	[0.503, 1.596]	[0.467, 1.560]	[0.070, 1.328]
3	[0.574, 1.571]	[0.605, 1.550]	[0.601, 1.527]	[0.604, 1.505]	[0.590, 1.481]	[0.569, 1.455]	[0.230, 1.283]
4	[0.619, 1.492]	[0.648, 1.471]	[0.654, 1.451]	[0.652,1.432]	[0.643, 1.413]	[0.628, 1.393]	[0.387, 1.254]
5	[0.652, 1.438]	[0.679, 1.418]	[0.686, 1.399]	[0.685, 1.383]	[0.679, 1.367]	[0.652, 1.333]	[0.485, 1.234]
6	[0.678, 1.398]	[0.703, 1.378]	[0.710, 1.362]	[0.710,1.347]	[0.706, 1.333]	[0.668, 1.305]	[0.550, 1.218]
7	[0.698, 1.368]	[0.722, 1.348]	[0.729, 1.333]	[0.730, 1.320]	[0.727, 1.307]	[0.697, 1.319]	[0.598, 1.206]
8	[0.715, 1.343]	[0.738, 1.324]	[0.745, 1.310]	[0.746, 1.298]	[0.744, 1.286]	[0.720, 1.294]	[0.634, 1.195]
9	[0.729, 1.323]	[0.751, 1.305]	[0.758, 1.291]	[0.760, 1.280]	[0.758, 1.269]	[0.753, 1.259]	[0.663, 1.187]
10	[0.741, 1.306]	[0.762, 1.288]	[0.769, 1.275]	[0.771, 1.264]	[0.770, 1.255]	[0.766, 1.245]	[0.686, 1.179]
15	[0.784, 1.248]	[0.802, 1.232]	[0.809, 1.222]	[0.811, 1.212]	[0.811, 1.206]	[0.808, 1.199]	[0.759, 1.152]
20	[0.811, 1.213]	[0.827, 1.199]	[0.833, 1.190]	[0.835, 1.184]	[0.835, 1.178]	[0.834, 1.172]	[0.779, 1.135]
50	[0.877, 1.133]	[0.887, 1.124]	[0.892, 1.118]	[0.894, 1.114]	[0.895, 1.111]	[0.895, 1.108]	[0.884, 1.091]

Similarly, the relative efficiency of  $\hat{\theta}_{MR}^{PT}$  with respect to  $\hat{\theta}_{MR}$  can be obtained as:

$$r.e \left( \hat{\theta}_{MR}^{PT}, \hat{\theta}_{MR} \right) = \{1 - r (H_{2r+4}(c_2\lambda) - H_{2r+4}(c_1\lambda)) + 2r (H_{2r+2}(c_2\lambda) - H_{2r+2}(c_1\lambda)) + (r + 1) (H_{2r}(c_2\lambda) - H_{2r}(c_1\lambda)) (\lambda^2 - \lambda)\}^{-1}.$$

Note that

$$\lim_{\lambda \rightarrow \infty} r.e \left( \hat{\theta}_{MR}^{PT}, \hat{\theta}_{MR} \right) = \lim_{\lambda \rightarrow \infty} r.e \left( \hat{\theta}_{ML}^{PT}, \hat{\theta}_{ML} \right) = 1, \\ \lim_{\lambda \rightarrow 0} r.e \left( \hat{\theta}_{MR}^{PT}, \hat{\theta}_{MR} \right) = \lim_{\lambda \rightarrow 0} r.e \left( \hat{\theta}_{ML}^{PT}, \hat{\theta}_{ML} \right) = 1.$$

Figures 1-3 shows the behavior of proposed estimators for different values of  $\alpha$  and  $r$ . We observe that the relative efficiencies of proposed estimators first decreases and get a minimum then it increases and reaches its maximum. Then it get down and after crossing the line 1 gets its minimum. Finally, as  $\lambda$  goes to infinity, the RE tends to the line 1. Also it can be observed that the maximum relative efficiencies are a decreasing function of  $\alpha$  while the minimum relative efficiencies are increasing function of  $\alpha$ . We see that the maximum relative efficiencies of the new proposed estimators are very higher than usual estimators. Moreover, the maximum relative efficiency of MRPTE is over the maximum relative efficiency of the MLPTE. Further, for the small values of  $r$  and large values of  $\alpha$ , the minimum relative of the MRPTE is lower than MLPTE. Also for the small vales of  $r$  the bounds that MLPTE dominate the MLE is larger than the bound that MRPTE outperform the MRE. So in spite of the fact that the MRE has minimum risk, in comparing to the MLE, we see the MRPTE is not uniformly better than the MLPTE. As a consequence, we recommend the users to conduct the MRPTE in a certain intervals to get more efficient estimators. Finally, as  $r$  increases, the proposed estimators behaves similarly. To be more specific, Tables 1 and 2 present the domination bounds of proposed MLPTE and MRPTE for different values of  $\alpha$  and  $r$  which is useful in practice.

**5. Summary and Conclusion** In this paper we considered two types of preliminary test estimators for the scale parameter of the exponential model based on Type II censored censored data. We especially observed that MRPTE has better performances than the MLPTE in the sense of having maximum relative efficiencies. however, it is not admissible estimator in general. Note that if the location parameter  $\mu$  is unknown then one can use the results of Baklizi. (2005)

TABLE 2.  $\hat{\theta}_{MR}^{PT}$  dominate  $\hat{\theta}_{MR}$  for different sample size and  $\alpha$  level

r	$\alpha$						
	0.01	0.05	0.10	0.15	0.20	0.25	0.50
2	[0.435,1.571]	[0.310, 1.543]	[0.123, 1.511]	[0.034 , 1.482]	[0.027, 1.454]	[0.025, 1.429]	[0.026, 1.339]
3	[0.531, 1.491]	[0.463, 1.461]	[0.355, 1.430]	[0.213, 1.401]	[0.081, 1.374]	[0.059, 1.350]	[0.052, 1.258]
4	[0.589, 1.437]	[0.547, 1.407]	[0.474, 1.377]	[0.378,1.350]	[ 0.252, 1.325]	[0.123, 1.302]	[0.081, 1.213]
5	[0.629, 1.397]	[0.601, 1.367]	[0.548, 1.339]	[0.477, 1.314]	[0.384, 1.291]	[0.260, 1.270]	[0.111, 1.184]
6	[0.659, 1.366]	[0.640, 1.337]	[0.599, 1.311]	[0.543,1.288]	[0.470, 1.266]	[0.373, 1.246]	[0.139, 1.164]
7	[0.682, 1.341]	[0.669, 1.313]	[0.636, 1.289]	[0.591, 1.267]	[0.532, 1.247]	[0.453, 1.228]	[0.166, 1.149]
8	[0.701, 1.321]	[0.692, 1.294]	[0.665, 1.270]	[0.628, 1.250]	[0.578, 1.232]	[0.512, 1.214]	[0.193, 1.138]
9	[0.717, 1.303]	[0.711, 1.277]	[0.689, 1.255]	[0.657, 1.237]	[0.614, 1.219]	[0.558, 1.202]	[0.218, 1.129]
10	[0.731, 1.289]	[0.727, 1.264]	[0.708, 1.243]	[0.680, 1.225]	[0.643, 1.208]	[0.593, 1.193]	[0.242, 1.122]
15	[0.778, 1.237]	[0.780, 1.216]	[0.770, 1.199]	[0.754, 1.185]	[0.732, 1.172]	[0.703, 1.160]	[0.359, 1.099]
20	[0.806, 1.206]	[0.810, 1.187]	[0.805, 1.173]	[0.794, 1.161]	[0.779, 1.151]	[0.759, 1.140]	[0.485, 1.086]
50	[0.875, 1.131]	[0.881, 1.119]	[0.881, 1.111]	[0.879, 1.104]	[0.875, 1.099]	[0.868, 1.093]	[0.779, 1.060]

for constructing the PTE mimicking the approach that present in this article . It should be mentioned here that when both of the parameters are unknown then the PTEs for the location and Scale parameters can be defined based on the generalized likelihood ration test (GLRT). This work is in progress by the authors and we hope that new results come out in near future. The findings of this paper can be applied in the estimation theory and practical situation when some suspected non-sample prior information are available.

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FIGURE 1. Relative Efficiencies of MLPTE and MRPTE

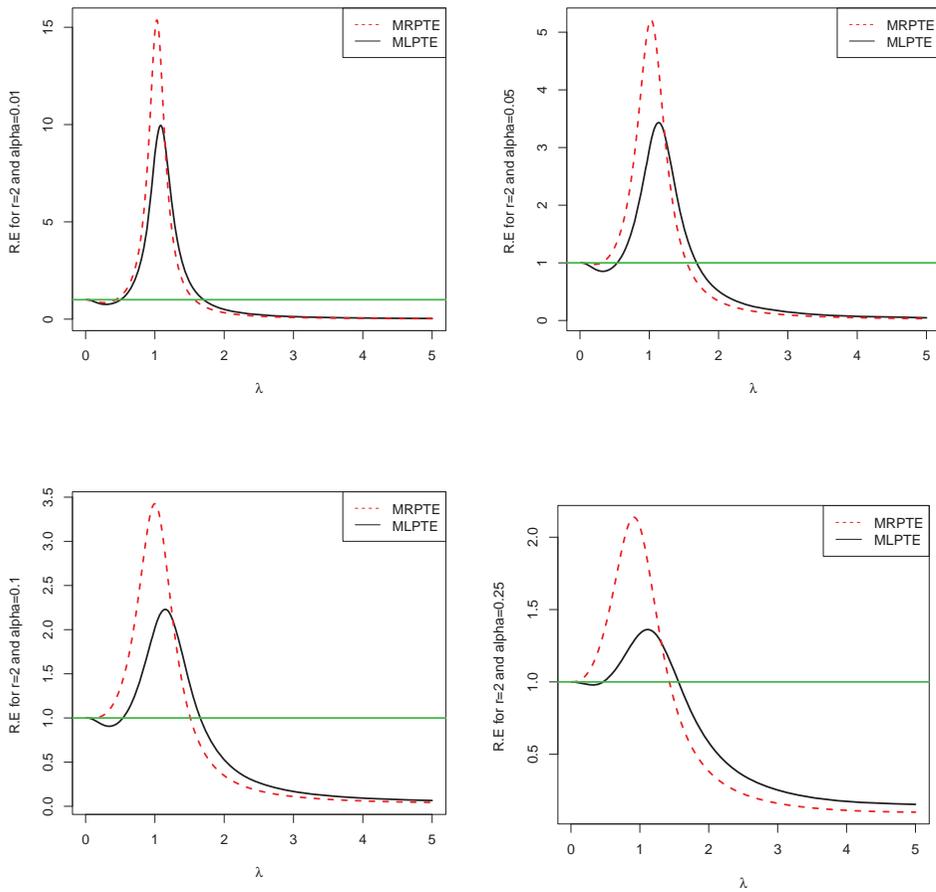


FIGURE 2. Relative Efficiencies of MLPTE and MRPTE

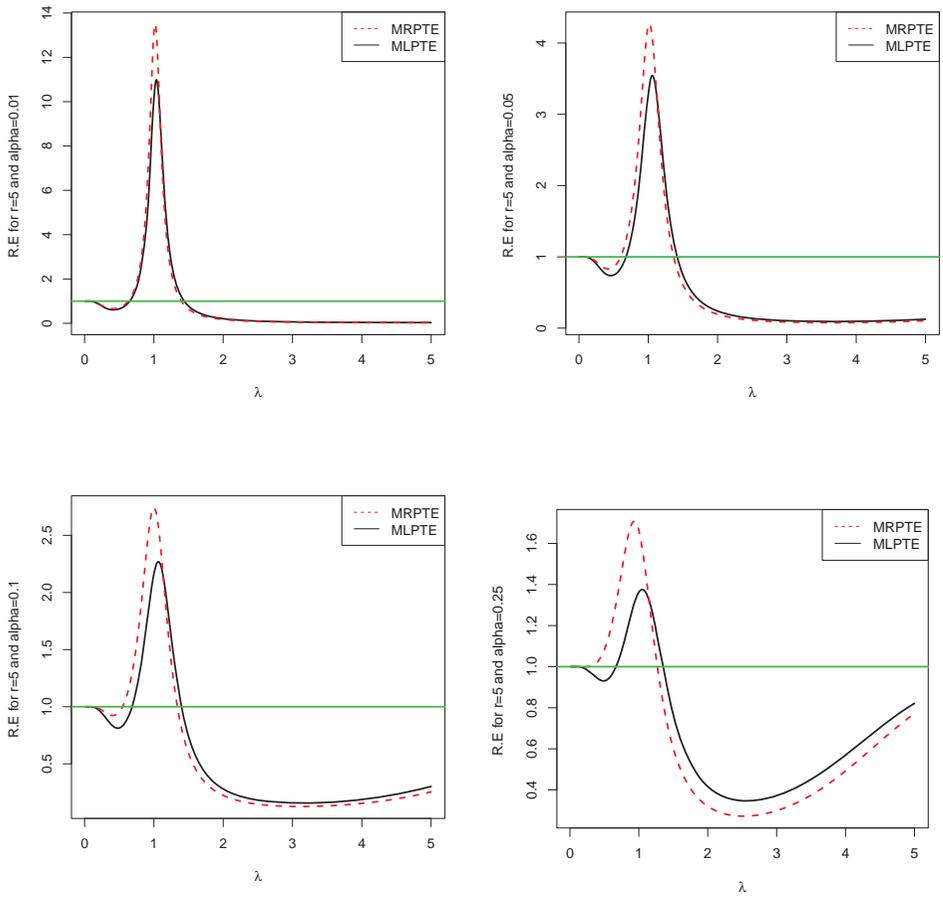
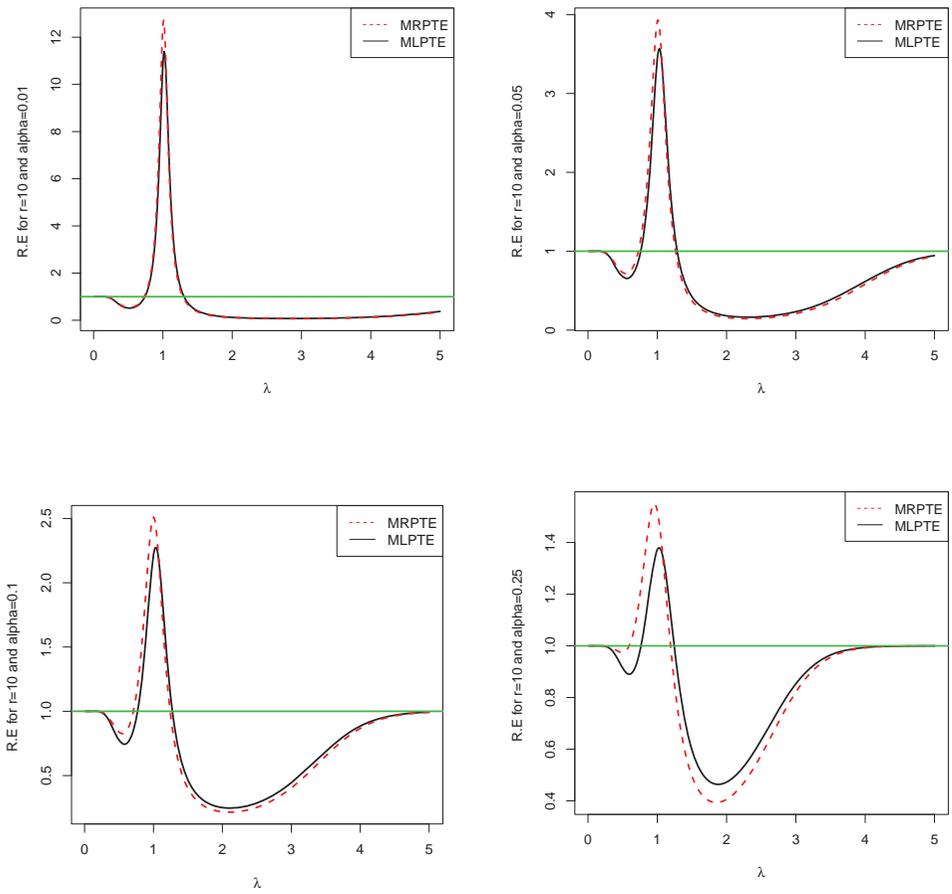


FIGURE 3. Relative Efficiencies of MLPTE and MRPTE



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