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ON PREINVEXITY FOR STOCHASTIC PROCESSES

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Abstract: In this paper, we introduce preinvex and invex stochastic processes, and we provide related well known Hermite-Hadamard integral inequality for preinvex stochastic processes by considering their left derivative, right derivative, and derivative processes.

Key words: convex stochastic processes; mean-square differentiable; mean square integral; Hermite-Hadamard inequality; preinvex functions; invex functions.

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1. Introduction

A stochastic process $\{X(t) : t \in I\}$ is a parameterized collection of random variables defined on a common probability space $(\Omega, \Im, P)$. Its parameter t is usually considered to be time. Then $X(t)$, which can also be shown as $X(t, \omega)$ for $\omega \in \Omega$, is considered to be state or position of the process at time t. For any fixed outcome ω of sample space Ω, the deterministic mapping $t \to X(t, \omega)$ denotes a realization, trajectory or sample path of the process. For any particular $t \in I$ the mapping depends ω alone, i.e., then we obtain a random variable. It can be said that, $X(t, \omega)$ *changes in time in a random manner*. We restrict our attention to continuous time stochastic processes, i.e., index set is $I = [0, \infty)$.

There are various ways to define stochastic monotonicity and convexity for stochastic processes, and it is of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities [12]. In [10] Nikodem proposed convex stochastic processes and gave some properties which are also known for classical convex functions. Temporal and spatiotemporal stochastic convexity was defined in [13] and [14], respectively for discrete time stochastic processes with illustrative examples. Convexity notions in sample path sense can also be found in [2], and the references therein. Jensen-convex, $\lambda-$convex, Wright-convex stochastic processes were introduced in [16], [17]. Time stochastic s-convexity was taken into account in [4] by using order preserving functions of majorizations. Kotrys [6] extended the classical Hermite-Hadamard inequality to convex stochastic processes. A class of the generalized convex stochastic processes, namely strongly convex stochastic processes was also proposed by Kotrys in [7]. The author's findings led to our motivation to build our work.

The well-known Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

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is used to provide estimations of the mean value of a continuous convex function $f:[a,b]\to\mathbb{R}$. In probabilistic point of view, 1.1 gives a lower bound and an upper bound for $E[f(X)]$ where X is uniformly distributed over the interval $[a,b]$[3]. In recent years, there has been an extensive interest in providing inequalities involving variety of convexity generalizations. Two of the significant generalizations are invex and preinvex functions introduced by Ben-Israel and Mond [1] and Hanson [5], respectively. Our goal in this paper is to establish Hermite-Hadamard inequality for another generalization of convex stochastic processes, namely preinvex stochastic processes.

2. Preliminaries

In this section we recall some basic definitions and notions about invex sets, preinvex and invex functions, additionally on continuity concepts and differentiability for stochastic processes, mean-square integral of a stochastic process.

Definition 1. A non-empty closed subset I of $\mathbb{R}^n$ is said to be invex set with respect to the given vector function $\eta: I\times I\longrightarrow \mathbb{R}^n$ (or $\eta-$invex, or $\eta-$connected set) if $u+\lambda\eta(v,u)\in I$ for all $u,v\in I$ and $\lambda\in[0,1]$.

Clearly, any convex set is an invex set with respect to $\eta(v,u)=v-u$. Geometrically, endpoints belonging to the set and line segment joining the endpoints are contained in a convex set. Convex sets cannot be disconnected, but invex sets can be disconnected. Definition 1 essentially says that there is a path starting from a point u which is contained in I. We do not require that the point v should be the one of endpoints of the path [11].

Definition 2. Let $I\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta: I\times I\longrightarrow \mathbb{R}^n$. Then the function (not necessarily differentiable) $f: I\to\mathbb{R}$ is said to be preinvex with respect to η if

$$f(u+\lambda\eta(v,u))\leq(1-\lambda)f(u)+\lambda f(v) \tag{2.1}$$

for each $u,v\in I$ and $\lambda\in[0,1]$.

Any convex function is preinvex with respect to $\eta(v,u)=v-u$, but the converse is not necessarily true.

Definition 3. For a differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be invex if there exists a vector function $\eta:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that

$$f(x)-f(u)\geq[\nabla f(u)]^T\eta(x,u) \tag{2.2}$$

for all $x,u\in\mathbb{R}^n$.

Any differentiable preinvex function is also an invex function [1]. An invex function may not be preinvex, $f(x)=\exp(x)$ is a counterexample, it is invex with respect to $\eta(x,u)=-1$, but not preinvex with respect to same η. Mohan and Neogy [9] proved that an invex function is also preinvex under following Condition C.

Condition C. Let $\eta: I\times I\longrightarrow\mathbb{R}^n$. It is told that the function η satisfies Condition C if,

(C1) $\eta(u,u+\lambda\eta(v,u))=-\lambda\eta(v,u)$

(C2) $\eta(v,u+\lambda\eta(v,u))=(1-\lambda)\eta(v,u)$

for all $u,v\in I$ and $\lambda\in[0,1]$.

Additionally, from condition C, we have

$$\eta(u+\lambda_2\eta(v,u),u+\lambda_1\eta(v,u))=(\lambda_2-\lambda_1)\eta(v,u) \tag{2.3}$$

for all $u,v\in I$ and $\lambda_1,\lambda_2\in[0,1]$. Note that, $\eta(u,u)=0$ for all $u\in I$. See [8] for more detail on preinvex and invex functions.

Definition 4. A real-valued stochastic process $\{X(t)|t\in I\}$ is said to be

(i) *continuous in probability* in I if

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)$$

(where $P-$lim denotes limit in probability) or equivalently

$$\lim_{t \to t_0} P\{|X(t, \cdot) - X(t_0, \cdot)| > \varepsilon\} = 0$$

for any arbitrary small enough $\varepsilon > 0$ and all $t_0 \in I$.
(ii) *mean-square continuous* (or *continuous in quadratic mean*) in I if

$$\lim_{t \to t_0} E[(X(t) - X(t_0))^2] = 0$$

such that $E[X(t)^2] < \infty$, for all $t_0 \in I$.
(iii) *mean-square differentiable* in I if it is mean square continuous and there exists a process $X^{'}(t, \cdot)$ ("speed" of the process) such that

$$\lim_{t \to t_0} E[\left(\frac{X(t) - X(t_0)}{t - t_0} - X'(t_0)\right)^2] = 0.$$

Different types of continuity concepts can be defined for stochastic processes. Surely (everywhere) and almost surely (almost everywhere or sample path) convergences are rarely used in applications due to restrictive requirement, that is, as $t \to t_0$, $X(t, \omega)$ has to approach $X(t_0, \omega)$ for each outcome $\omega \in S \subseteq \Omega$. For further reading on stochastic calculus, reader may refer to [15].

Definition 5. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t)^2] < \infty$ for all $t \in I$. Let $[a, b] \subset I, a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ arbitrarily for $k = 1, \ldots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t)$ on $[a, b]$ if the following identity holds:

$$\lim_{n \to \infty} E[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2] = 0. \tag{2.4}$$

Then we can write

$$\int_a^b X(t, \cdot)dt = Y(\cdot) \ (a.e.).$$

Mean square integral operator is increasing, that is,

$$\int_a^b X(t, \cdot)dt \leq \int_a^b Z(t, \cdot)dt \ (a.e.)$$

where $X(t, \cdot) \leq Z(t, \cdot)$ *(a.e.)* in $[a, b]$.

Throughout this paper, we assume that $I \subseteq [0, \infty)$ is a η-invex interval and the function η satisfies Condition C.

3. Hermite-Hadamard Inequality For Preinvex Stochastic Processes

In this section, we propose the important generalizations of convexity for stochastic processes which are called preinvex and invex stochastic processes. Furthermore, Hermite-Hadamard inequality is extended for preinvex stochastic processes.

Two random variables can be compared in the sense of stochastic dominance or stochastic order by using their expectations, distribution functions, likelihood ratios, hazard rates, or majorization, etc. Motivated by the definition of convexity for random processes [10] we generalize the idea of preinvexity to processes. Convexity definitions were also given for random variables in [12].

Definition 6. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process (not necessarily mean-square differentiable) on η-invex index set I. $X(t,\cdot)$ is called preinvex with respect to η if

$$X(u+\lambda\eta(v,u),\cdot) \le (1-\lambda)X(u,\cdot)+\lambda X(v,\cdot)(a.e.) \tag{3.1}$$

for all $u,v \in I$ and $\lambda \in [0,1]$. For a preinvex stochastic process, the inequality 3.1 holds almost everywhere on Ω, i.e., almost every sample path of X will be a preinvex function.

For instance, the convex stochastic process $X : (0,1)\times(0,1) \to \mathbb{R}$ defined by

$$X(t,\omega) = \begin{cases} t^2, \text{ when } t \ne \omega \\ 0, \text{ when } t = \omega \end{cases}$$

is a preinvex stochastic process with respect to $\eta(v,u) = -u$.
In 3.1, if λ is fixed number in (0,1), then X is called
(i) $\lambda-$preinvex stochastic process
(ii) mid-preinvex stochastic process for $\lambda = \frac{1}{2}$.
If we choose $\eta(v,u) = v-u$, then preinvex $X(t,\cdot)$ is also a convex stochastic process, that is, class of convex stochastic processes is contained by the class of preinvex stochastic processes.

Definition 7. Let $X : I \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process. Then $X(t,\cdot)$ is called invex with respect to η if

$$X(t,\cdot)-X(t_0,\cdot) \ge X'(t_0,\cdot)\eta(t,t_0)(a.e.) \tag{3.2}$$

for all $t,t_0 \in I$.

If X is a mean square differentiable stochastic process, then it is also mean square continuous from Definition 4. Mean square continuity guarantees continuity in probability, that is, as $t \to t_0$,

$$P\left\{\left|\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0} - X'(t_0)\right| > \varepsilon\right\} \le \frac{E[\left(\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0} - X'(t_0)\right)^2]}{\varepsilon^2} \to 0$$

for any small enough $\varepsilon > 0$ and all $t_0 \in I$.

For a preinvex stochastic process, from now on, let us assume that η is skew-symmetric, i.e., $\eta(t,t_0) = -\eta(t_0,t)$ for all $t,t_0 \in intI$, and $\eta(t,t_0) \ge 0$ for such $t \ge t_0$. In this instance, the right and left derivative processes can be defined as:

$$X'_+(t_0,\cdot) = P - \lim_{t\to t_0^+} \frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0} = P - \lim_{\substack{t \ge t_0 \\ t_0+\eta(t,t_0) \le t}} \frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0}, \tag{3.3}$$

and

$$X'_{-}(t_0,\cdot) = P - \lim_{t\to t_0^-}\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0} = P - \lim_{\substack{t\le t_0\\ t+\eta(t_0,t)\le t_0}}\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0}, \tag{3.4}$$

respectively.

Our purpose is to adapt the Hermite-Hadamard inequality for preinvex processes. In order to establish our argument, we give, prove, and utilize the following Lemma 1 and Lemma 2.

LEMMA 1. *Let $X : I\times\Omega\to\mathbb{R}$ be a preinvex stochastic process and fulfill any of the following conditions:*
(i) its right derivative process exists at $t_0\in I$,
(ii) X is decreasing, and its left derivative process exists at $t_0\in I$,
(iii) X is mean-square differentiable at $t_0\in I$
then, there exists a random variable $A:\Omega\to\mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)\eta(t,t_0)+X(t_0,\cdot)$ that is

$$X(t,\cdot)\ge A(\cdot)\eta(t,t_0)+X(t_0,\cdot)(a.e.) \tag{3.5}$$

for all $t\in I$.

PROOF. *First, let us investigate by supposing that only (i) holds. Taking the limit of $X(t,\cdot)$ as $t\to t_0^+$, since $t_0+\eta(t,t_0)\le t$, we obtain*

$$X(t,\cdot)-X(t_0,\cdot)\overset{P}{=}X'_{+}(t_0,\cdot)(t-t_0)\ge X'_{+}(t_0,\cdot)\eta(t,t_0).$$

Here $\overset{P}{=}$ stands for "limit in probability".

Now let the condition (ii) be satisfied and $t\le t_0-\eta(t_0,t)=t_0+\eta(t,t_0)\le t_0$. Since X is decreasing, and its left derivative process exists at t_0, we get

$$X(t,\cdot)\ge X(t_0-\eta(t_0,t),\cdot)\overset{P}{=}X'_{-}(t_0,\cdot)\eta(t,t_0)+X(t_0,\cdot).$$

Finally, X possesses invexity, because X is a mean-square differentiable preinvex stochastic process as defined in condition (iii). Thus, the proof of Lemma 1 is finished by using the inequality 3.2.

LEMMA 2. *If the stochastic process $X : I\times\Omega\to\mathbb{R}$ has the form $X(t,\cdot)=A(\cdot)\eta(t,u)+B(\cdot)$ where $A(\cdot),B(\cdot)$ are random variables such that $E[A^2],E[B^2]<\infty$ and $[u,u+\eta(v,u)]\subset I$, then*

$$\int_u^{u+\eta(v,u)} X(t,\cdot)dt = A(\cdot)\frac{\eta^2(v,u)}{2}+B(\cdot)\eta(v,u)\ (a.e.). \tag{3.6}$$

PROOF. *By dividing the interval $[u,u+\eta(v,u)]$ into n subintervals each of equal length and choosing midpoints of the subintervals as points in the partition, we yield:*

$$\lim_{n\to\infty}E[\left(\sum_{k=1}^{n}X(\Theta_k)(t_k-t_{k-1})-\left(A\frac{\eta^2(v,u)}{2}+B\eta(v,u)\right)\right)^2]$$
$$=\lim_{n\to\infty}E[\left(\sum_{k=1}^{n}\left(A\eta(\Theta_k,u)+B\right)(t_k-t_{k-1})-\left(A\frac{\eta^2(v,u)}{2}+B\eta(v,u)\right)\right)^2]$$

$$= \lim_{n\to\infty} E[\left(A \left(\sum_{k=1}^{n} \eta(\Theta_k, u)(t_k - t_{k-1}) - \frac{\eta^2(v,u)}{2} \right) + B \left(\sum_{k=1}^{n} (t_k - t_{k-1}) - \eta(v,u) \right) \right)^2]$$
$$= E[A^2] \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{2k-1}{2n} \eta(v,u)(t_k - t_{k-1}) - \frac{\eta^2(v,u)}{2} \right)^2 = 0.$$

THEOREM 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a preinvex stochastic process under any of the assumptions of Lemma 1, then for any $u, v \in I$ we have*

$$X\left(\frac{2u+\eta(v,u)}{2}, \cdot\right) \le \frac{1}{\eta(v,u)} \int_{u}^{u+\eta(v,u)} X(t,\cdot)dt \le \frac{X(u,\cdot)+X(v,\cdot)}{2} \ (a.e.). \tag{3.7}$$

PROOF. *First, we show the right-hand side of 3.7 by putting $t = u + \lambda\eta(v,u)$. Since η satisfies Condition C, from 2.3 it appears that*

$$\eta(t,u) = \eta(u+\lambda\eta(v,u),u) = \lambda\eta(v,u). \tag{3.8}$$

Since X is preinvex, using 3.8 we get

$$X(t,\cdot) \le (1-\lambda)X(u,\cdot) + \lambda X(v,\cdot) = \frac{1}{\eta(v,u)} \left[\eta(t,u)\left(X(v,\cdot) - X(u,\cdot)\right) + \eta(v,u)X(u,\cdot)\right] \ (a.e.). \tag{3.9}$$

Taking into account 3.9 in 3.6, it is easy to see that the right-hand side of 3.7 can be established as follows:

$$\int_{u}^{u+\eta(v,u)} X(t,\cdot)dt \le \frac{1}{\eta(v,u)} \int_{u}^{u+\eta(v,u)} \left[\eta(t,u)\underbrace{(X(v,\cdot) - X(u,\cdot))}_{A(\cdot)} + \eta(v,u)\underbrace{X(u,\cdot)}_{B(\cdot)} \right] dt$$
$$= \eta(v,u)\frac{X(u,\cdot)+X(v,\cdot)}{2} (a.e.). \tag{3.10}$$

Now, let us obtain the left-hand side of 3.7. Under the assumptions of Lemma 1, there exists a random variable $A : \Omega \to \mathbb{R}$ at

$$t_0 = \frac{2u+\eta(v,u)}{2}$$

such that

$$X(t,\cdot) \ge A(\cdot)\eta\left(t, \frac{2u+\eta(v,u)}{2}\right) + X\left(\frac{2u+\eta(v,u)}{2}, \cdot\right) \tag{3.11}$$

Thus, putting $t = u + \lambda\eta(v,u)$ from 3.8 we get

$$\eta\left(t, \frac{2u+\eta(v,u)}{2}\right) = \left(\lambda - \frac{1}{2}\right)\eta(v,u) = \eta(t,u) - \frac{\eta(v,u)}{2}. \tag{3.12}$$

Now, we turn to compute the integral (in the mean square sense) as follow:

$$\int_{u}^{u+\eta(v,u)} X(t,\cdot)dt \ge \int_{u}^{u+\eta(v,u)} \left[A(\cdot)\eta\left(t, \frac{2u+\eta(v,u)}{2}\right) + X\left(\frac{2u+\eta(v,u)}{2}, \cdot\right) \right] dt$$

$$= \int_{u}^{u+\eta(v,u)} \left[A(\cdot) \left(\eta(t,u) - \frac{\eta(v,u)}{2} \right) + X(\frac{2u+\eta(v,u)}{2}, \cdot) \right] dt$$
$$= \int_{u}^{u+\eta(v,u)} \left[A(\cdot)\eta(t,u) - \frac{1}{2}A(\cdot)\eta(v,u) + X(\frac{2u+\eta(v,u)}{2}, \cdot) \right] dt$$
$$= A(\cdot)\frac{\eta^2(v,u)}{2} - \frac{1}{2}A(\cdot)\eta(v,u)\eta(v,u) + X(\frac{2u+\eta(v,u)}{2}, \cdot)\eta(v,u)$$
$$= X(\frac{2u+\eta(v,u)}{2}, \cdot)\eta(v,u) \ (a.e.).$$

As a result, the proof of Theorem 1 is completed.

4. Conclusion

In this paper, we propose the important generalization of convexity for stochastic processes which are called preinvex and invex stochastic processes. These concepts are particularly interesting from optimization view point, since it provides a broader setting to study the optimization and mathematical programming problems. In convex programming, the set of feasible solutions is convex, a local minimum is also a global minimum, i.e., the Karush-Kuhn-Tucker necessary optimality conditions are sufficient. We also obtain a Hermite-Hadamard inequality for preinvex stochastic processes under some suitable conditions. As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes. Applying some type of inequalities for stochastic processes is another promising direction for future research.

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