

SOME PROPERTIES AND APPLICATIONS OF SHIFTED PROPORTIONAL STOCHASTIC ORDERS

Jarrahiferiz, J., Mohtashami Borzadaran, G. R.* and Rezaei Roknabadi, A. H.

Department of Statistics
School of Mathematical Sciences
Ferdowsi University of Mashhad
Mashhad-IRAN

Abstract: The purpose of this paper is to study shifted versions of the proportional likelihood ratio order, proportional (reversed) hazard rate order and some related aging classes. We give some properties and relationships to other stochastic comparisons which are known in the literature and we study some applications in the reliability theory. Furthermore, conditions for preservation of this orderings under weighted distributions are given.

Key words: Up (Down) proportional likelihood ratio order, Up (Down) proportional hazard rate order, Preservation, Shock models, Coherent system.

History: Submitted: 27 June 2012; Revised: 9 August 2013; Accepted: 13 August 2013

1. Introduction

Stochastic orders have been shown that are very useful in applied probability, statistics, reliability, operation research, economics and other fields. Various types of stochastic orders and associated properties have been developed rapidly over the years. A lot of research works have been done on likelihood ratio, hazard rate and reversed hazard rate orders due to their properties and applications in the various sciences. For example, the hazard rate order which is well-known and it is a useful tool in reliability theory and reversed hazard rate order is defined via stochastic comparison of inactivity time. We can refer the interested readers to Muller (1997), Kijima (1998), Chandra and Roy (2001), Gupta and Nanda (2001), Hu and Zhu (2001), Nanda and Shaked (2001), Kochar et al. (2002), Kayid and Ahmad (2004), Ahmad et al. (2005) and Shaked and Shanthikumar (2007). Ramos-Romero and Sordo-Diaz (2001) introduced a new stochastic order between two continuous and non-negative random variables and called it proportional likelihood ratio (plr) order, which is closely related to the usual likelihood ratio order. The proportional likelihood ratio order can be used to characterize random variables whose have log-concave (log-convex) densities. Furthermore, using stochastic comparisons of random variable X to itself according to the proportional stochastic orderings, new classes of life time distributions is generated which, in turn, are useful to describe aging process of a lifetime system. It is shown that some well-known distributions namely truncated normal, exponential, power series and beta distributions belong to those classes which make them of increasing interest. So, they studied increasing proportional likelihood ratio class (IPLR) and as an application, they showed that the IPLR properties is sufficient conditions for the Lorenz ordering of truncated distributions.

Belzunce et al. (2002), extended hazard rate and reversed hazard rate orders to proportional state in the same manner and called them proportional hazard rate and proportional reversed hazard rate orders respectively. So, they studied their properties, preservations and relations with other

* Corresponding author. E-mail address: grmohtashami@um.ac.ir (Mohtashami Borzadaran, G. R.)

orders.

Shifted stochastic orders are useful tools for establishing interesting inequalities and have been introduced in papers such as Shanthikumar and Yao (1986), Nakai (1995), Brown and Shanthikumar (1998) and Belzunce et al. (2001). Lillo et al. (2001) studied in details four shifted stochastic orders, namely the up (down) likelihood ratio order and the up (down) hazard rate order. Recently, Aboukalam and Kayid (2007) obtained some new results about shifted hazard rate and shifted likelihood ratio orders.

Note that the usual stochastic ordering is equivalent to its shifted and proportional versions. In general, the shifted and proportional versions are stronger orderings and easy to verify in many situations, so they are helpful to check what components are more reliable, and consequently systems formed from them.

In this paper, we study the shifted proportional likelihood ratio and shifted proportional hazard rate orders and some results. Also, conditions for preservation of this orderings under weighted distributions are given and finally some applications of them in shock models and reliability systems are studied.

2. Preliminaries

Let X and Y be two continuous and non-negative random variables with densities f and g , distribution functions F and G , hazard rate functions r_F and r_G and reversed hazard rate functions \tilde{r}_F and \tilde{r}_G , each with an interval support S_X and S_Y respectively. Denote by l_X the left endpoint and by u_X the right endpoint of S_X . Similarly, define l_Y and u_Y for Y . The values u_X and u_Y may be infinite. Throughout the article, we will use the term *increasing* in place of *non-decreasing* and *decreasing* in place of *non-increasing* and we take $a/0$ to be ∞ , whenever $a > 0$, also, let $\lambda \leq 1$ be any positive constant.

DEFINITION 1. Let X and Y be two continuous and non-negative random variables. It is said that

- (1) X is smaller than Y in the usual stochastic order ($X \leq_{st} Y$), if $P(X > x) \leq P(Y > x)$, $\forall x \in (-\infty, \infty)$.
- (2) X is smaller than Y in the likelihood ratio order ($X \leq_{lr} Y$), if $\frac{g(x)}{f(x)}$ increases in x over the $S_X \cup S_Y$.
- (3) X is smaller than Y in the hazard rate order ($X \leq_{hr} Y$), if $r_F(x) \geq r_G(x)$, $\forall x \in (-\infty, \infty)$.
- (4) X is smaller than Y in the reversed hazard rate order ($X \leq_{rh} Y$), if $\tilde{r}_F(x) \leq \tilde{r}_G(x)$, $\forall x \in (-\infty, \infty)$.
- (5) X is smaller than Y in the up likelihood ratio order (up hazard rate order, up reversed hazard rate order), denoted by $X \leq_{lr\uparrow} (\leq_{hr\uparrow}, \leq_{rh\uparrow})Y$, if $X - x \leq_{lr} (\leq_{hr}, \leq_{rh})Y$, $\forall x \geq 0$.
- (6) X is smaller than Y in the down likelihood ratio order (down hazard rate order, down reversed hazard rate order), denoted by $X \leq_{lr\downarrow} (\leq_{hr\downarrow}, \leq_{rh\downarrow})Y$, if $X \leq_{lr} (\leq_{hr}, \leq_{rh})[Y - x | Y > x]$, $\forall x \geq 0$.

The proportional likelihood ratio order and its related aging classes have been introduced by Ramos-Romero and Sordo-Diaz (2001). Then Belzunce et al. (2002) presented proportional hazard rate and proportional reversed hazard rate orders and their related aging classes similarly and studied preservation properties of them under some system structures.

DEFINITION 2. If X and Y are continuous and non-negative random variables, then we will say that

(1) X is smaller than Y in the proportional likelihood ratio order (proportional hazard rate order, proportional reversed hazard rate order), denoted by, $X \leq_{plr} (\leq_{phr}, \leq_{prh})Y$, if $\lambda X \leq_{lr} (\leq_{hr}, \leq_{rh})Y$ for all $0 < \lambda \leq 1$.

(2) X has the increasing proportional likelihood ratio (increasing proportional hazard rate) property, $X \in IPLR (IPHR)$, if $X \leq_{plr} (\leq_{phr})X$.

In the next section we introduce shifted proportional likelihood ratio order and study some results.

3. The shifted proportional likelihood ratio order

Up (Down) shifted proportional likelihood ratio order is introduced for continuous and non-negative random variables. Here, we obtain some results for the up shifted proportional likelihood ratio order. The results can be obtained for the down shifted proportional likelihood ratio order in a similar manner.

DEFINITION 3. Let X and Y be two continuous and non-negative random variables.

(1) We say that X is smaller than Y in the up proportional likelihood ratio order ($X \leq_{plr\uparrow} Y$), if $[X - x | X > x] \leq_{plr} Y$ for all $x \geq 0$.

(2) X is smaller than Y in the down proportional likelihood ratio order ($X \leq_{plr\downarrow} Y$), if $X \leq_{plr} [Y - x | Y > x]$ for all $x \geq 0$.

THEOREM 1. Let X and Y be two continuous and non-negative random variables.

(1) $X \leq_{plr\uparrow} Y$ if and only if $\frac{g(\lambda t)}{f(t+x)}$ is increasing in $t \in (l_X - x, u_X - x) \cup (\frac{l_Y}{\lambda}, \frac{u_Y}{\lambda})$ for all $x \geq 0$.

(2) $X \leq_{plr\downarrow} Y$ if and only if $\frac{g(\lambda t+x)}{f(t)}$ is increasing in $t \geq 0$, for all $x \geq 0$.

(3) $X \leq_{plr\uparrow} Y$ and $X \leq_{plr\downarrow} Y$ imply $X \leq_{lr\uparrow} Y$ and $X \leq_{lr\downarrow} Y$ respectively.

DEFINITION 4. A continuous non-negative random variable X admits up increasing proportional likelihood ratio property, denoted by $X \in UIPLR$, if $X \leq_{plr\uparrow} X$.

THEOREM 2. Let X and Y be two continuous non-negative random variables. Then,

(1) If $X \leq_{lr\uparrow} Y$ and $X \in IPLR$ then, $X \leq_{plr\uparrow} Y$.

(2) If $X \leq_{plr} Y$ and f is log-concave then, $X \leq_{plr\uparrow} Y$.

(3) If $X \leq_{lr} Y$ and $Y \in UIPLR$ then, $X \leq_{plr\uparrow} Y$.

Proof We prove only the case (3), the proofs of the other parts are clear. When $X \leq_{lr} Y$, then $l_X \leq l_Y$ and $u_X \leq u_Y$, so, $\frac{l_X}{\lambda} \leq \frac{l_Y}{\lambda}$ and $\frac{u_X}{\lambda} \leq \frac{u_Y}{\lambda}$. If $u_X - x \leq \frac{l_Y}{\lambda}$, the proof is complete. Otherwise, if $\frac{l_Y}{\lambda} < u_X - x$, then for each $t \in (\frac{l_Y}{\lambda}, u_X - x)$ we get $\frac{g(\lambda t)}{f(t+x)} = \frac{g(\lambda t)}{g(t+x)} \frac{g(t+x)}{f(t+x)}$, which both right fractions are increasing in $t \in (\frac{l_Y}{\lambda}, u_X - x)$, note that if $t \in (\frac{l_Y}{\lambda} - x, \frac{u_Y}{\lambda} - x)$, then $g(t+x) > 0$. Since $l_X - x \leq \frac{l_Y}{\lambda} < u_X - x \leq \frac{u_Y}{\lambda}$, using Definition 3 the proof is complete. \square

The following result helps us to find for systems the components with less dispersion in up proportional likelihood ratio order.

THEOREM 3. Let X and Y be two continuous non-negative random variables. If $X \leq_{plr\uparrow} Y$, then, there exists a random variable $Z \in UIPLR$, such that $X \leq_{plr} Z \leq_{plr} Y$.

Proof For $u_X \leq l_Y$ take Z as an arbitrary random variable with UIPLR property taking values on $[u_X, l_Y]$. Suppose that $l_Y \leq u_X$. Set $k_X = \frac{f'}{f}$ and $k_Y = \frac{g'}{g}$ which are transformations of X and Y , respectively. We know that

$$\begin{aligned} X \leq_{plr\uparrow} Y &\iff k_X(t+x) \leq \lambda k_Y(\lambda t), \quad \forall x \geq 0, t \in (l_Y, u_X - x) \\ &\iff k_X(t') \leq \lambda k_Y(\lambda t), \quad \forall l_Y \leq t < t' \leq u_Y. \end{aligned} \tag{3.1}$$

Define $k^*(t) = \max_{\nu \geq t} k_X(\nu)$, $t \in (l_Y, u_X)$. By Lillo et al. (2001), $f^*(t) = (\int_{l_Y}^{u_X} e^{\int_t^s k^*(\nu) d\nu} ds)^{-1}$, $t \in (l_Y, u_X)$, is a density function on (l_Y, u_X) , which admits the UIPLR property. Hence, $k_X(t) \leq \lambda k_Y^*(\lambda t)$, $\forall t \in (l_Y, u_X)$, thus $X \leq_{plr} Z$. Finally, from (3.1), $k^*(t) \leq \lambda k_Y(\lambda t)$, $\forall t \in (l_Y, u_X)$, so $Z \leq_{plr} Y$ holds. \square

EXAMPLE 1. Let X and Y be two independent random variables. If $X \leq_{plr\uparrow} Y$, using Corollary 1.C.34 of Shaked and Shanthikumar (2007), there exists a random variable $Z \in UIPLR$ such that $\min\{X, Y\} \leq_{plr} \min\{Y, Z\}$ and $\max\{X, Y\} \leq_{plr} \max\{Y, Z\}$.

THEOREM 4. Let $\{X_j\}$ and $\{Y_j\}$ be two sequences of continuous and non-negative random variables such that $X_j \rightarrow X$ in law and $Y_j \rightarrow Y$ in law as $j \rightarrow \infty$, where X and Y are continuous and non-negative random variables. If $X_j \leq_{plr\uparrow} Y_j$, $j = 1, 2, \dots$ then, $X \leq_{plr\uparrow} Y$.

Proof For each interval $I \in [0, \infty)$ write $P(I) = P(X \in I)$, $P_j(I) = P(X_j \in I)$, $Q(I) = P(Y \in I)$ and $Q_j(I) = P(Y_j \in I)$. Also, for any interval $I \in [0, \infty]$ and $x > 0$ define $I+x = \{y+x; y \in I\}$. Finally, for any two intervals I and J we say $I \leq J$ if $x \in I$ and $y \in J$ imply $x \leq y$.

We see that $X_j \leq_{plr\uparrow} Y_j$ if and only if $P_j(J+x)Q_j(\lambda I) \leq P_j(I+x)Q_j(J\lambda)$ for all intervals $I, J \in [0, \infty]$ such that $I \leq J$ and $x > 0$. Since $X_j \rightarrow X$ in law and $Y_j \rightarrow Y$ in law as $j \rightarrow \infty$, we get, using the continuity of X and Y , that $P(J+x)Q(\lambda I) \leq P(I+x)Q(J\lambda)$ for all intervals $I, J \in [0, \infty]$ such that $I \leq J$ and $x > 0$, that leads to, $X \leq_{plr\uparrow} Y$. \square

The weighted distributions were first proposed by Fisher (1934) and formalized later by Rao (1965). Statistical applications of the weighted distributions are especially used to the analysis of data relating to human populations, ecology, forestry and reliability. Readers may refer to Patil and Rao (1977), Gupta and Keating (1986), Kochar and Gupta (1987), Jain et al. (1989), Bartoszewicz and Skolimowska (2004), Belzunce et al. (2004), Misra et al. (2008), Bartoszewicz (2009) and Izadkhah et al. (2013) for more details. Here, we present the results for up proportional likelihood ratio order, for others, they can be obtained similarly.

Let $w_i : R \rightarrow R^+, i = 1, 2$ be a weight function such that $0 < E(w_i(X_i)) < \infty, i = 1, 2$. Define a random variable X^{w_i} with distribution function

$$F_{w_i}(x) = \frac{1}{E(w_i(X))} \int_{-\infty}^x w_i(u) dF(u),$$

and we call X^{w_i} the weighted random variable corresponding to X and w_i for $i = 1, 2$. For an absolutely continuous distribution F the probability density function of X^{w_i} is given by

$$f_{w_i}(x) = \frac{w_i(x)f(x)}{E(w_i(X))}, \quad i = 1, 2.$$

Define by $S_X^{w_i} = \{x \in S_{X_i} : w_i(x) > 0\}$ the support of X^{w_i} where S_{X_i} is the support of $X_i, i = 1, 2$. We now list some well-known weight functions as follows:

- (a) size biased of order $p_i, i = 1, 2; w_i(x) = x^{p_i}, p_i = 1, 2, \dots, i = 1, 2;$
 (b) moment generating; $w_i(x) = e^{l_i x}, -\infty < l_i < +\infty, i = 1, 2;$
 (c) right truncated at $\theta_i, i = 1, 2; w_i(x) = I_{(-\infty, \theta_i]}(x), -\infty < \theta_i < +\infty, i = 1, 2;$
 (d) left truncated at $\theta_i, i = 1, 2; w_i(x) = I_{(\theta_i, \infty)}(x), -\infty < \theta_i < +\infty, i = 1, 2;$
 (e) proportional hazard rates; $w_i(x) = (\bar{F}_i(x))^{k_i-1}, k_i > 0, i = 1, 2;$
 (f) proportional reversed hazard rates; $w_i(x) = (F_i(x))^{j_i-1}, j_i > 0, i = 1, 2;$
 (g) order statistics; $w_i(x) = (F_i(x))^{k_i-1} \cdot (\bar{F}_i(x))^{n_i-k_i}, k_i = 1, 2, \dots, n_i, i = 1, 2;$
 (h) upper records; $w_i(x) = (-\ln \bar{F}_i(x))^{n_i}, n_i = 1, 2, \dots, i = 1, 2;$
 (i) lower records; $w_i(x) = (-\ln F_i(x))^{m_i}, m_i = 1, 2, \dots, i = 1, 2;$
 (j) general weight function; $w_i(x) = e^{l_i x} x^{p_i} F_i^{j_i}(x) \bar{F}_i^{k_i}(x)$ where $l_i, p_i, j_i, k_i \in R, i = 1, 2.$

The general weight function, given in (j), contains the ones in (a), (b), (e), (f), and (g), respectively, when $(l_i = j_i = k_i = 0), (j_i = k_i = p_i = 0), (l_i = j_i = p_i = 0, k_i = k_i - 1), (l_i = k_i = p_i = 0, j_i = j_i - 1),$ and $(l_i = p_i = 0, j_i = k_i - 1, k_i = n_i - k_i)$ for $i = 1, 2.$

In the next theorems, conditions for preservation of this orderings under weighted distributions are given.

THEOREM 5. *If $X \in UIPLR$ and $w(\lambda x)$ is increasing in x then, $X \leq_{plr\uparrow} X^w.$*

For example some weights such as $w(x) = x^i, w(x) = F^n(x)$ and $w(x) = e^{lx}$ satisfy in the above result.

EXAMPLE 2. Let $X \sim Exp(\mu)$ and $w(x) = x^i$, then $X^w \sim gamma(i + 1, \mu)$ and $\frac{f_w(\lambda x+t)}{f(x)}$ is increasing and so $X \leq_{plr\uparrow} X^w.$

THEOREM 6. *Suppose that w_1 and w_2 be weight functions and X^{w_1} and X^{w_2} be corresponding random variables with densities functions f_{w_1} and f_{w_2} respectively. if $X \in UIPLR$ and $\frac{w_2(\lambda t)}{w_1(t+x)}$ is increasing in t , then, $X^{w_1} \leq_{plr\uparrow} X^{w_2}.$*

Proof We get

$$\begin{aligned} \frac{f_{w_2}(\lambda t)}{f_{w_1}(t+x)} &= \left[\frac{\lambda w_2(\lambda t) f(\lambda t)}{E(w_2(X))} \right] \left[\frac{E(w_1(X))}{w_1(t+x) f(t+x)} \right] \\ &= \lambda \left[\frac{E(w_1(X))}{E(w_2(X))} \right] \left[\frac{w_2(\lambda t)}{w_1(t+x)} \right] \left[\frac{f(\lambda t)}{f(t+x)} \right], \end{aligned}$$

which is increasing in $t.$ \square

EXAMPLE 3. Let $X \in UIPLR$, with consideration the following weight functions we have $X^{w_1} \leq_{plr\uparrow} X^{w_2}.$

(1) If $w_1(t) = w_2(t) = t$ then, $\frac{w_2(\lambda t)}{w_1(t+x)} = \frac{\lambda t}{t+x}$ in which $\frac{d}{dt} \frac{\lambda t}{t+x} \geq 0.$

(2) If $w_1(t) = t^n$ and $w_2(t) = t^m$ for $t \geq 0$ and $n > m$, then, $h(t) = \frac{w_2(\lambda t)}{w_1(t+x)} = \frac{(\lambda t)^n}{(t+x)^m}$ and,

$$\begin{aligned} \frac{d}{dt} h(t) &= \frac{n\lambda^n t^{n-1} (t+x)^m - m(\lambda t)^n (t+x)^{m-1}}{(t+x)^{2m}} \\ &= \frac{\lambda^n t^{n-1} (t+x)^{m-1} [n(t+x) - mt]}{(t+x)^{2m}} \\ &= \frac{\lambda^n t^{n-1} (t+x)^{m-1} [t(n-m) + nx]}{(t+x)^{2m}} \geq 0. \end{aligned}$$

(3) If $w_1(t) = e^{l_1 t}$ and $w_2(t) = e^{l_2 t}$ such that $\frac{l_1}{l_2} < \lambda.$ For other weights it can be checked similarly.

4. The shifted proportional hazard rate order

We can do a process similar to the previous section for proportional (reversed) hazard rate order. Here, we do it only for proportional hazard rate order. It is evident that all of the results can be obtained for proportional reversed hazard rate order similarly.

DEFINITION 5. For continuous and non-negative random variables X and Y , we say that

(1) X is smaller than Y in the up proportional hazard rate order ($X \leq_{phr\uparrow} Y$), If $[X - x|X > x] \leq_{phr} Y$, $\forall x \geq 0$.

(2) X is smaller than Y in the down proportional hazard rate order ($X \leq_{phr\downarrow} Y$), if $X \leq_{phr} [Y - x|Y > x]$, $\forall x \geq 0$.

Actually, $X \leq_{phr\uparrow} Y$ if and only if, $\frac{\bar{G}(\lambda t)}{F(t+x)}$, $\forall x \geq 0$, is increasing in $t \in (0, \frac{u_Y}{\lambda})$ and $X \leq_{phr\downarrow} Y$ if and only if, $\frac{\bar{G}(\lambda t+x)}{F(t)}$, $\forall x \geq 0$ is increasing in $t \geq 0$.

(3) X has up increasing proportional hazard rate property, $X \in UIPHR$, if $X \leq_{phr\uparrow} X$, and has down increasing proportional hazard rate property, $X \in DIPHR$, if $X \leq_{phr\downarrow} X$. It is evident that, $X \leq_{phr\uparrow} Y \Rightarrow X \leq_{phr} Y$ and $X \leq_{hr\uparrow} Y$.

THEOREM 7. Let X and Y be two continuous and non-negative random variables.

(1) If $X \leq_{hr} Y$ and $Y \in UIPHR$ ($DIPHR$), then $X \leq_{phr\uparrow} Y$ ($X \leq_{phr\downarrow} Y$).

(2) If $X \leq_{hr\uparrow} Y$ and \bar{F} is log-concave then $X \leq_{phr\uparrow} Y$.

THEOREM 8. Let X and Y be two continuous and non-negative random variables. If $X \leq_{phr\uparrow} Y$, then for each concave and strictly increasing function ψ that $\psi(Y) \in UIPHR$ we have $\psi(X) \leq_{phr\uparrow} \psi(Y)$.

Proof Let $\bar{F}_{\psi(X)}$ and $\bar{G}_{\psi(Y)}$ be survival functions of $\psi(X)$ and $\psi(Y)$, respectively. We should prove that $\frac{\bar{G}_{\psi(Y)}(\lambda t)}{\bar{F}_{\psi(X)}(t+x)}$ is increasing in t . We know that,

$$\frac{\bar{G}_{\psi(Y)}(\lambda t)}{\bar{F}_{\psi(X)}(t+x)} = \frac{\bar{G}(\psi^{-1}(\lambda t))}{\bar{F}(\psi^{-1}(t+x))} = \left[\frac{\bar{G}(\psi^{-1}(\lambda t))}{\bar{G}(\psi^{-1}(t))} \right] \left[\frac{\bar{G}(\psi^{-1}(t))}{\bar{F}(\psi^{-1}(t+x))} \right].$$

By assuming $\psi(Y) \in UIPHR$, so, $\frac{\bar{G}(\psi^{-1}(\lambda t))}{\bar{G}(\psi^{-1}(t))}$ is increasing in t . Also, when $X \leq_{phr\uparrow} Y$, using Theorem 6.24 of Lillo et al. (2001), $\frac{\bar{G}(\psi^{-1}(t))}{\bar{F}(\psi^{-1}(t+x))}$ is increasing in t , so, $\frac{\bar{G}_{\psi(Y)}(t)}{\bar{F}_{\psi(X)}(t+x)}$ is increasing in t , consequently, $\psi(X) \leq_{phr\uparrow} \psi(Y)$. \square

THEOREM 9. Let X and Y be two continuous and non-negative random variables which $X \leq_{phr\downarrow} Y$ and ψ is strictly increasing function such that $\psi(Y) \in DIPHR$, then, $\psi(X) \leq_{phr\downarrow} \psi(Y)$.

Proof Suppose that $\bar{F}_{\psi(X)}$ and $\bar{G}_{\psi(Y)}$ be survival functions of $\psi(X)$ and $\psi(Y)$ respectively. We must prove that $\frac{\bar{G}_{\psi(Y)}(\lambda t+x)}{\bar{F}_{\psi(X)}(t)}$ is increasing in t . We have,

$$\frac{\bar{G}_{\psi(Y)}(\lambda t+x)}{\bar{F}_{\psi(X)}(t)} = \frac{\bar{G}(\psi^{-1}(\lambda t+x))}{\bar{F}(\psi^{-1}(t))} = \left[\frac{\bar{G}(\psi^{-1}(\lambda t+x))}{\bar{G}(\psi^{-1}(t))} \right] \left[\frac{\bar{G}(\psi^{-1}(t))}{\bar{F}(\psi^{-1}(t))} \right],$$

when $\psi(Y) \in \text{DIPHR}$, $\frac{\bar{G}(\psi^{-1}(\lambda t+x))}{\bar{G}(\psi^{-1}(t))}$ is increasing in t . Since $X \leq_{\text{phr}\downarrow} Y \implies X \leq_{\text{hr}} Y$ and so $\psi(X) \leq_{\text{hr}} \psi(Y)$, then, $\frac{\bar{G}(\psi^{-1}(t))}{F(\psi^{-1}(t))}$ is increasing in t . Hence, $\frac{\bar{G}_{\psi(Y)}(\lambda t+x)}{F_{\psi(X)}(t)}$ is increasing in t , that is, $\psi(X) \leq_{\text{phr}\downarrow} \psi(Y)$. \square

THEOREM 10. Let X and Y be continuous and non-negative random variables. If $X \leq_{\text{phr}\uparrow} Y$, then, there exists a random variable $Z \in \text{UIPHR}$, such that $X \leq_{\text{phr}} Z \leq_{\text{phr}} Y$.

Proof If $u_X \leq l_Y$ then take Z to be any random variables that has UIPHR property on $[u_X, l_Y]$. Therefore, suppose that $l_Y \leq u_X$.

$$\begin{aligned} X \leq_{\text{phr}\uparrow} Y &\iff r_X(t+x) \geq \lambda r_Y(\lambda t), \quad \forall x \geq 0, t \in (l_Y, u_X - x) \\ &\iff r_X(t') \geq \lambda r_Y(\lambda t), \quad l_Y \leq t < t' \leq u_X. \end{aligned} \quad (4.1)$$

Define $r^*(t) = \max_{\nu \leq t} r_Y(\nu)$, $t \in (l_Y, u_Y)$. By Lillo et al. (2001), r^* defines a hazard rate function on (l_Y, u_Y) and it is sufficient that consider Z has the hazard rate function r^* . By assumption r^* is increasing in t , so, $r^*(t') \geq \lambda r^*(\lambda t)$, and therefore, $Z \leq_{\text{phr}} Y$. Finally, from (4.1), it follows that $r_X(t) \geq \lambda r^*(\lambda t)$, for all $t \in (l_Y, u_X)$, so, $X \leq_{\text{phr}} Z$. \square

5. Applications

In the Poisson shock models, if N and X_j denote the number of shocks survived by the system and the random interval time between the $j-1$ -th and j -th shocks respectively, then the lifetime T of the system is given by $T = \sum_{j=1}^N X_j$. In particular, if the intervals are assumed to be independent and exponentially distributed with parameter μ , then the distribution function of T can be written as

$$H(t) = \sum_{k=0}^{\infty} \frac{e^{-\mu t} (\mu t)^k}{k!} P_k, \quad t \geq 0,$$

where $P_k = P(N \leq k)$ for all positive integer k .

Suppose that a device is subject to shocks occurring randomly as events in a Poisson process with constant μ . Also, consider that the device has probability $\bar{P}_k = 1 - P_k$ of surviving the first k shocks, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \dots$. The survival function of the device is given by

$$\bar{F}(t) = \sum_{k=0}^{\infty} \frac{e^{-\mu t} (\mu t)^k}{k!} \bar{P}_k. \quad (5.1)$$

Let $p_{k+1} = \bar{P}_k - \bar{P}_{k+1}$, $k = 0, 1, 2, \dots$, and let $f(t)$ be the probability function corresponding to survival function $\bar{F}(t)$, such that

$$f(t) = \sum_{k=0}^{\infty} \frac{e^{-\mu t} (\mu t)^k}{k!} \mu p_{k+1}. \quad (5.2)$$

Consider another device that has contains similar to the above device and has probability of surviving first k shocks, where $1 = \bar{Q}_0 \geq \bar{Q}_1 \geq \dots$. The survival function of this device is given by

$$\bar{G}(t) = \sum_{k=0}^{\infty} \frac{e^{-\mu t} (\mu t)^k}{k!} \bar{Q}_k. \quad (5.3)$$

Let $q_{k+1} = \bar{Q}_k - \bar{Q}_{k+1}$, $k = 0, 1, 2, \dots$, and let $g(t)$ be the probability function corresponding to survival function $\bar{G}(t)$, such that

$$g(t) = \sum_{k=0}^{\infty} \frac{e^{-\mu t} (\mu t)^k}{k!} \mu q_{k+1}. \quad (5.4)$$

Notice that if N is distributed as geometric distribution with p.m.f $p_j = q^{j-1}p$, $j = 1, 2, \dots$ then the distribution given in (5.1), coincides with an exponential distribution with parameter $\mu(1-p)$. Using this, we obtain in the next theorem the optimal upper bound for lifetime of the system.

THEOREM 11. *Let N_1 and N_2 denote the number of shocks survived by two systems with distribution functions defined as in (5.1) and (5.3), respectively. Assume further that G_p denotes the geometric distribution with parameter p , which is independent of N_2 . Then, $N_1 \leq_{hr\uparrow} \min(G_p, N_2)$, for all $p \in (0, 1]$, implies that $X \leq_{phr\uparrow} Y$.*

Proof For any $c > 0$ and $\lambda \in (0, 1]$, we get

$$\begin{aligned} \bar{G}(\lambda t + s) - c\bar{F}(t) &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} (\lambda^{k+j} \bar{Q}_{k+j} - c_1 \bar{P}_k) \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} (P(N_2 > k+j)P(G_{1-\lambda} > k+j) - c_1 \bar{P}_k) \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} (P(\min(N_2, G_{1-\lambda}) > k+j) - c_1 \bar{P}_k), \end{aligned}$$

where $c_1 = ce^{-\mu t(1-\lambda)}$. Now, by assumption $P(\min(N_2, G_{1-\lambda}) > k+j) - c_1 \bar{P}_k$ has at most one change of sign. The conclusion now follows. \square

The following result is also obtained for $\leq_{plr\uparrow}$ ordering.

THEOREM 12. *Under the assumptions of Theorem 11 if, $N_1 \leq_{lr\uparrow} (\frac{N_2+G_p}{2} | N_2 = G_p)$, for all $p \in (0, 1]$, then $X \leq_{plr\uparrow} Y$.*

Proof For any real c and $\lambda \in (0, 1]$, we have

$$\begin{aligned} g(\lambda t + s) - cf(t) &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} \mu (\lambda^{k+j} q_{k+j+1} - c_1 p_{k+1}) \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} \mu \left(\frac{P(G_{1-\lambda} = k+j+1)P(N_2 = k+j+1)}{1-\lambda} - c_1 p_{k+1} \right) \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} \mu \left(\frac{P(N_2 = G_{1-\lambda})}{1-\lambda} P\left(\frac{N_2 + G_{1-\lambda}}{2} = k+j+1 | N_2 = G_{1-\lambda}\right) - c_1 p_{k+1} \right) \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda s} s^j}{j!} \sum_{k=0}^{\infty} \frac{e^{-\mu\lambda t} (\mu t)^k}{k!} \mu \frac{P(N_2 = G_{1-\lambda})}{1-\lambda} \left(P\left(\frac{N_2 + G_{1-\lambda}}{2} = k+j+1 | N_2 = G_{1-\lambda}\right) - c_2 p_{k+1} \right), \end{aligned}$$

where $c_1 = ce^{-\mu t(1-\lambda)}$ and $c_2 = \frac{(1-\lambda)c_1}{P(N_2 = G_{1-\lambda})}$. By assumption $P\left(\frac{N_2 + G_{1-\lambda}}{2} = k+j+1 | N_2 = G_{1-\lambda}\right) - c_2 p_{k+1}$ has at most one change of sign. Hence the proof is obtained. \square

Consider two coherent systems C_1 and C_2 , each consisting of n iid components. Suppose that the lifetime of components from C_1 and C_2 have distribution functions F and G respectively. In the following result, the preservation of up proportional hazard rate order is proved for a coherent system with *i.i.d.* components, that is an analogous of Theorem 3.1 of Aboukalam and Kayid (2007).

Remark Let X and Y be non-negative and absolutely continuous random variables. If $X \leq_{phr\uparrow} Y$ then, $\bar{G}(t+x) \leq \bar{F}(\lambda x)$.

THEOREM 13. Let $h(p)$ be the reliability function of coherent system of n independent and identical components having first and second derivatives $h'(p)$ and $h''(p)$ respectively. If $\frac{ph''(p)}{h'(p)}$ is decreasing and $\bar{G} \leq_{phr\uparrow} \bar{F}$, then, $h(\bar{G}) \leq_{phr\uparrow} h(\bar{F})$.

Proof We must show that, $\frac{\lambda f(\lambda t)h'(\bar{F}(\lambda t))}{g(t+x)h'(\bar{G}(t+x))}$ is increasing in $t > 0$, equivalently,

$$\lambda[\lambda f'(\lambda t)g(t+x) - g'(t+x)f(\lambda t)] + \lambda[f(\lambda t)g(t+x)] \left[\frac{g(t+x)h''(\bar{G}(t+x))}{h'(\bar{G}(t+x))} - \frac{\lambda f(\lambda t)h''(\bar{F}(\lambda t))}{h'(\bar{F}(\lambda t))} \right] \geq 0,$$

thus,

$$\begin{aligned} & \lambda[\lambda f'(\lambda t)g(t+x) - g'(t+x)f(\lambda t)] \\ & + \lambda[f(\lambda t)g(t+x)] \left[\frac{g(t+x)}{\bar{G}(t+x)} \frac{\bar{G}(t+x)h''(\bar{G}(t+x))}{h'(\bar{G}(t+x))} - \frac{\lambda f(\lambda t)}{\bar{F}(\lambda t)} \frac{\bar{F}(\lambda t)h''(\bar{F}(\lambda t))}{h'(\bar{F}(\lambda t))} \right] \geq 0, \end{aligned} \quad (5.5)$$

which is non-negative because the both terms are non-negative by assumption. \square

Consider a system of n independent and not necessarily identical components in which the i th component has survival function $\bar{F}_i(t) = 1 - F_i(t)$, $i = 1, 2, \dots, n$. Let $h(\mathbf{P}) = h(p_1, p_2, \dots, p_n)$ be the system reliability function. In the following theorem we compare the random lifetimes of two systems according to the up proportional hazard rate order.

THEOREM 14. If $\sum_{i=1}^n \frac{p_i \partial h / \partial p_i}{h(\mathbf{P})}$ is decreasing in p_i and $X_i \leq_{phr\uparrow} Y$, for all $i = 1, 2, \dots, n$ then, $h(\mathbf{X}) \leq_{phr\uparrow} h(\mathbf{Y})$.

Proof We know that the hazard rate function of a coherent system is:

$$r_{h(\mathbf{X})}(z+t) = \sum_{i=1}^n r_{X_i}(z+t) \bar{F}_{X_i}(z+t) \frac{\partial h / \partial p_i}{h(\mathbf{P})} \Big|_{p_i = \bar{F}_{X_i}(z+t)}$$

Hence, hypothesis $X_i \leq_{phr\uparrow} Y$, gives,

$$r_{h(\mathbf{X})}(z+t) \geq \lambda r_Y(\lambda z) \sum_{i=1}^n \bar{F}_{X_i}(z+t) \frac{\partial h / \partial p_i}{h(\mathbf{P})} \Big|_{p_i = \bar{F}_{X_i}(z+t)},$$

using the assumption implies that,

$$r_{h(\mathbf{X})}(z+t) \geq \lambda r_Y(\lambda z) \sum_{i=1}^n \bar{F}_Y(\lambda z) \frac{\partial h / \partial p_i}{h(\mathbf{P})} \Big|_{p_i = \bar{F}_{X_i}(z+t)} \equiv \lambda r_{h(\mathbf{Y})}(\lambda z),$$

and the proof is complete. \square

In the next theorem we obtain the preservation of the up proportional hazard rate order under the formation of coherent systems with different and i.i.d. components. Similar results hold concerning the shifted reversed proportional hazard rate orderings.

THEOREM 15. Let $h_1(p)$ and $h_2(p)$ be the reliability functions of two coherent systems with n and m components respectively, such that

$$\frac{h'_1(p)}{h_1(p)} \geq \frac{h'_2(p)}{h_2(p)} \text{ for all } p \in [0, 1] \quad (5.6)$$

and

$$p \frac{h'(p)}{h(p)} \text{ is decreasing in } p, \text{ when } h = h_1 \text{ or } h = h_2. \quad (5.7)$$

If $X \leq_{phr\uparrow} Y$, then $T_{h_1(X)} \leq_{hr\uparrow} T_{h_2(X)}$ where $T_{h_1(X)}$ and $T_{h_2(X)}$ are the lifetimes of coherent systems.

Proof From Definition 5, it is sufficient to prove that $r_{h_1(X)}(t+x) \geq \lambda q_{h_2(Y)}(\lambda t)$ for all $t \geq 0$ and $x \geq 0$, that is

$$\begin{aligned} r_{h_1(X)}(t+x) &= \frac{f(t+x) \bar{F}(t+x) h'_1(\bar{F}(t+x))}{\bar{F}(t+x) h_1(\bar{F}(t+x))} \\ &\geq \lambda \frac{g(\lambda t) \bar{G}(\lambda t) h'_2(\bar{G}(\lambda t))}{\bar{G}(\lambda t) h_2(\bar{G}(\lambda t))} = \lambda q_{h_2(Y)}(\lambda t), \end{aligned}$$

which holds by the hypothesis $X \leq_{phr\uparrow} Y$ and the conditions (5.6) and (5.7). \square

Remark that if we replace conditions (5.6) and (5.7) with

$$\frac{h'_1(p)}{1-h_1(p)} \geq \frac{h'_2(p)}{1-h_2(p)} \text{ for all } p \in [0, 1] \quad (5.8)$$

and

$$\frac{(1-p)h'(p)}{1-h(p)} \text{ is increasing in } p, \text{ when } h = h_1 \text{ or } h = h_2, \quad (5.9)$$

similar results hold for the up proportional reversed hazard rate ordering.

Now, we consider a set of independent and not necessarily identically distributed components with lifetime (X_1, \dots, X_n) and we establish comparisons in the up proportional hazard rate order between two coherent systems with different structures formed from this set of components. we will write $\mathbf{a}_{|s|}$ to denote the s -dimensional vector (a_1, a_2, \dots, a_s) .

THEOREM 16. Let X_1, \dots, X_n be the lifetimes of n independent components with increasing density functions, and $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{p}_{|m|})$ the reliability functions of two coherent systems with $n \geq m$. If

$$\frac{1}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \geq \frac{1}{h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \text{ for all } i = 1, \dots, m, \quad (5.10)$$

and

$$\frac{1}{h(\mathbf{p})} \frac{\partial h}{\partial p_i}(\mathbf{p}) \text{ is decreasing in each } p_s \text{ of } \mathbf{p} \text{ for all } i, \text{ when } h = h_1 \text{ or } h_2 \quad (5.11)$$

then $T_{h_1(X_{|n|})} = T_{h_1(X_1, \dots, X_n)} \leq_{phr\uparrow} T_{h_2(X_1, \dots, X_n)} = T_{h_2(X_{|n|})}$.

Proof The hazard rate function $r_{h(X)}$ of a coherent system with n components and reliability function $h(p_1, \dots, p_n)$ is given by

$$r_{h(X)}(t) = \sum_{i=1}^n f_i(t) \left[\frac{1}{h(\mathbf{p})} \frac{\partial h}{\partial p_i}(\mathbf{p}) \right]_{\mathbf{p}=(\bar{F}_1(t), \dots, \bar{F}_n(t))}, \quad (5.12)$$

- [11] Fisher, R. A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *Annals Eugenics*, 6, 13-25.
- [12] Gupta, R. C. and Keating, J. P. (1986). Relations for reliability measures under length biased sampling. *Statistical Papers*, 13, 49-56.
- [13] Gupta, R. D. and Nanda, A. K. (2001). Some results on (reversed) hazard rate ordering. *Communications in Statistics-Theory and Methods*, 30, 2447-2458.
- [14] Hu, T. and Zhu, Z. (2001). An analytic proof of the preservation of the up-shifted likelihood ratio order under convolutions. *Stochastic Processes and their Applications*, 95, 55-61.
- [15] Izadkhah, S., Rezaei Roknabadi, A. H., and Mohtashami Borzadaran, G. R. (2013). On Properties of Reversed Mean Residual Life Order for Weighted Distributions. *Communications in Statistics-Theory and Methods*, 42(5), 838-851.
- [16] Jain, K., Singh, H. and Bagai, I. (1989). Relations for reliability measures of weighted distributions. *Communications in Statistics-Theory and Methods*, 18, 4393-4412.
- [17] Kayid, M. and Ahmad, I. A. (2004). On the mean inactivity time ordering with reliability applications. *Probability in the Engineering and Informational Sciences*, 18, 395-409.
- [18] Kijima, M. (1998). Hazard rate and reversed hazard rate monotonicities in continuous-time Markov chains. *Journal of Applied Probability*, 35, 545-556.
- [19] Kochar, S. C. and Gupta, R. P. (1987). Some results on weighted distributions for positive-valued random variables. *Probability in the Engineering and Informational Sciences.*, 1, 417-423.
- [20] Kochar, S. C., Li, X. and Shaked, M. (2002). The total time on test transform and the excess wealth stochastic orders of distributions. *Advances in Applied Probability*, 34, 826-845.
- [21] Lillo, R. E., Nanda, A. K. and Shaked, M. (2001). Preservation of some likelihood ratio stochastic orders by order statistics. *Statistics and Probability Letters*, 51, 111-119.
- [22] Misra, N., Gupta, N. and Dhariyal, I. (2008). Preservation of some aging properties and stochastic orders by weighted distributions. *Communications in Statistics-Theory and Methods*, 37, 627-644.
- [23] Muller, A. (1997). Stochastic order generated by integrals: A unified approach. *Advances in Applied Probability*, 29, 414-428.
- [24] Nakai, T. (1995). A partially observable decision problem under a shifted likelihood ratio ordering. *Mathematical and Computer Modelling*, 22, 237-246.
- [25] Nanda, A. K. and Shaked, M. (2001). The hazard rate and reversed hazard rate orders with applications to order statistics. *Annals of the Institute of Statistical Mathematics*, Vol. 53, No. 4, 853-864.
- [26] Patil, G. P. and Rao, C. R. (1977). *The weighted distributions: A survey and their applications*. Applications of Statistics, Amsterdam: North Holland, 383-405.
- [27] Ramos-Romero, H. M. and Sordo-Diaz, M. A. (2001). The proportional likelihood ratio order and applications. *Questiio*, 25, 211-223.
- [28] Rao, C. R. (1965). *On discrete distributions arising out of methods of ascertainment*. Pergamon Press, Oxford and Statistical Publishing Society, 320-332.
- [29] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Academic Press, New York.
- [30] Shanthikumar, J. G. and Yao, D. D. (1986). The preservation of likelihood ratio ordering under convolutions. *Stochastic Processes and their Applications*, 23, 259-267.

