İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION
Vol. 6, No. 1, January 2013, pp. 42–55
ISSN 1300-4077 | 13 | 1 | 42 | 55

İSTATİSTİK

PERFORMANCE EVALUATION OF A FINITE BUFFER SYSTEM WITH VARYING RATES OF IMPATIENCE

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Abstract: In the analysis of reneging behavior, one approach is to assume random patience time. Each customer is assumed to follow identical distribution of patience time. However, there are many real life queuing systems where this assumption is not satisfied. Customers waiting in the system are often aware of their position in it and hence the reneging rate varies with the position of the customer in the system. This paper is an attempt to model such a reneging phenomena along with balking. Explicit closed form expressions of a number of performance measures have been presented. A numerical example with design aspects has also been presented to demonstrate the results derived.

Key words: Balking, Finite buffer queue, Impatience, Queuing, Reneging.
History: Submitted: 21 May 2012; Revised: 5 October 2012; Accepted: 14 November 2012

1. Introduction

In this competitive world, the key to survival of service oriented organizations rest in positive customer experience at the point of service delivery. It is argued that unless the customer is satisfied with the quality of service offered, customer loyalty may suffer. It is in this context that waiting for service attains importance, as queues are unavoidable. That being so, customers get impatient on having to experience a queue. This impatience finds reflection in two ways viz: balking and reneging. Even though these concepts have been dealt with in literature, closed form expressions are not always available. This paper is an attempt to fill this gap in literature.

By balking, we mean the phenomenon of customers arriving for service into a non-empty queue and leaving without joining the queue. There is no balking from an empty queue. Haight (1957) has provided a rationale, which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency, so that a queue of certain length will not be joined, to indifference where a non-zero queue is also joined.

A customer will be said to have reneged if after joining the system it gets impatient and leaves the system without receiving service. On joining the system, it has a patience time such that in case service is unavailable within this patience time, the customer reneges.

Reneging can be of two types-viz. reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). A customer can renege only as long as it is in the queue and we call this as reneging of type R_BOS. It cannot renege once it begins receiving service. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service gets started i.e. hair cut begins, the customer cannot leave till hair cutting is over. On the other hand, if customers can renege not only while waiting in queue but also while receiving service, we call such behavior as reneging of type R_EOS. "Ward and Bambos (2003) detailed a case of web system. In such a system, request for specific pages

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arrive at random points in time and are served in the order received. Impatient users may cancel their request at any time and that may occur either during service or before service has begun. Overall such dynamics arise in any queuing situation with time sensitive jobs, where reneging may occur both in queues and at the server".

In the analysis of reneging phenomena, one approach is to assume that each customer has a Markovian patience time, the distribution of which is position independent. However, it is our common day observation that there are systems where the customer is aware of its position in the system. For example customers queuing at the O.P.D. (out patient department) clinic of a hospital would know of their position in the queue. This invariably causes waiting customers to have higher rates of reneging in case their position in the queue is towards the end. It is not unreasonable then to expect that such customers who are positioned at a distance from the service facility have reneging rates, which are higher than reneging rates of customers who are near the service facility. In other words, we assume that customers are "position aware" and in this paper we model the reneging phenomenon in such a manner that the Markovian reneging rate is a function of the position of the customer in the system. Customers at higher states will be assumed to have higher reneging rates.

The subsequent sections of this paper are structured as follows. Section 2 contains a brief review of the literature. Section 3 and section 4 contains the derivation of steady state probabilities and performance measures respectively. We perform sensitivity analysis in section 5. A numerical example is discussed in section 6. Concluding statements are presented in section 7. The appendix contains some derivation.

2. Literature Survey

Barrer (1957a) carried out one of the early work on reneging where he considered deterministic reneging with single server Markovian arrival and service rates. Customers were selected randomly for service. In his subsequent work, Barrer (1957b) also considered deterministic reneging (of both R_BOS and R_EOS type) in a multi server scenario with FCFS discipline. The general method of solution was extended to two related queuing problems. Another early work was by Haight (1959). Ancher and Gafarian (1963) carried out an early work on Markovian reneging with Markovian arrival and service pattern. Kok and Tijms (1985) considered a single server queuing system where a customer becomes a lost customer when its service has not begun within a fixed time. Haghighi et al. (1986) considered a Markovian multiserver queuing model with balking as well as reneging. Each customer had a balking probability which was independent of the state of the system. Reneging discipline considered by them was R_BOS. Liu et al. (1987) considered an infinite server Markovian queuing system with reneging of type R_BOS. Customers had a choice of individual service or batch service, batch service being preferred by the customer. Brandt and Brandt (1999) considered a S-server system with two FCFS queues, where the arrival rates at the queues and the service may depend on number of customers 'n' being in service or in the first queue. Customers in the first queue were assumed impatient customers with deterministic reneging. Boots and Tijms (1999) considered an $M/M/C$ queue in which a customer leaves the system when its service has not begun within a fixed interval after its arrival. They have given the probabilistic proof of 'loss probability', which was expressed in a simple formula involving the waiting time probabilities in the standard $M/M/C$ queue. Ke and Wang (1999) considered the machine repair problem in which failed machines balk with probability $(1-b)$ and renege according to a negative exponential distribution. Another work using the concepts of balking and reneging in machine interference queue has been carried out by Al-Seedy and Al-Ibraheem (2001). Choi et al. (2001) introduced a simple approach for the analysis of the $M/M/C$ queue with a single class of customers and constant patience time by finding simple Markov process. Applying this approach, they analyzed the M/M/1 queue with two classes of customer in which class 1 customer have impatience of constant duration and class 2 customers have no impatience and lower priority than class 1 customers. Performance measures of both $M/M/C$ and $M/M/1$ queues were discussed. Zhang et al. (2005) considered an $M/M/1/N$ framework with Markovian reneging where they derived the steady state probabilities and formulated a cost model. Some performance measures were also discussed. Choudhury (2008) provided a detailed and lucid derivation of the distribution of virtual waiting time in a single server Markovian queuing system under R-BOS.

El- Paoumy (2008a) derived the analytical solution of $M^x/M/2/N$ queue for batch arrival system with Markovian reneging. Another paper on Markovian reneging was by Altman and Yechiali (2008). They derived the probability generating function of number of customers present in the system and some performance measures were discussed. Xiong and Altiok (2009) have provided approximations for some performance measures of a multi server queue with Poisson arrivals, general service time distribution and deterministic reneging.

Yechiali (2007) considered a multi server Markovian queue suffering occasional disaster breakdown. During such breakdown, all customers in the system are cleared. New arrivals during the breakdown period have an exponentially distributed patience time, such that if the service is not reactivated during this patience time, the customer reneges.

Other attempts at modeling reneging phenomenon include those by Baccelli et al. (1984), Martin and Artalejo (1995), Shawky (1997), Choi et al. (2004), Singh et al. (2007), El- Sherbiny (2008) and El-Paoumy and Ismail (2009) etc.

An early work on balking was carried out by Haight (1957). Jouini et al. (2008) modeled a call center as an M/M/s+M queue with endogenized customer reactions to announcements. They assumed that customers react by balking upon hearing the delay announcement and may subsequently renege if their realized waiting time exceeds the delay that has originally announced to them. They calculated the waiting time distribution i.e. announcement coverage and subsequent performance in terms of balking and reneging. Al-Seedy et al. (2009) presented an analysis for the M/M/c queue with balking and reneging. They assumed that arriving customers balked with a fixed probability and reneged according to a negative exponential distribution. The generating function technique was used to obtain the transient solution of system those results in a simple differential equation. Yue et al. (2009) considered an M/M/2 queuing system with balking and two heterogeneous servers, server 1 and server 2. They assumed that customers arrived according to a Poisson process and form a single waiting line where two parallel servers provided heterogeneous exponential service on a first-come first-served basis. It is also assumed that server 1 is perfectly reliable and server 2 is subject to breakdowns. They obtained the stationary condition where the system reaches a steady state and derived the steady state probabilities in a matrix form by using matrix-geometric solution method. They produced explicit expressions of some performance measures such as the mean system size, the average balking rate and the probabilities that server 2 is in various states. Choudhury and Medhi (2011) analyzed a multiserver Markovian queuing system under the assumption that customers may balk as well as renege. They assumed that each arriving customer has probability $(1-p)$ of balking from a system with no idle servers and for reneging they assumed that each customer joining the system have a random patience time. Explicit closed form expressions were presented. A numerical example with design aspects was also discussed to demonstrate results derived.

Some other papers which have considered both balking and reneging are the work by Shawky and El-Paoumy (2000), El- Paoumy (2008a and 2008b), El- Sherbiny (2008), Shawky and El-Paoumy (2008), Pazgal et al. (2008).

3. The Model and System State Probabilities

The model we deal with in this paper is the traditional M/M/1/k model with the restriction that customers may balk from a non-empty queue as well as may renege after they join the queue.

We shall assume that the inter arrival and service rates are λ and μ respectively. Analysis of M/M/1/k queuing model assumes significance from the fact that in the classical M/M/1 model, "it is assumed that the system can accommodate any number of units. In practice, this may seldom be the case. We have thus to consider the situation such that the system has limited waiting space and can hold a maximum number of k units (including the one being served)" {Medhi (1994)}. Our formulation differs from that of Medhi (1994) in that his model did not consider balking and reneging. As stated earlier we impose the additional restrictions of state dependent balking and position dependent reneging.

For balking it will be assumed that if the customer on arrival observes the system to be in state 'i', the probability that he will balk is 'i/k', $i = 1, 2, \ldots, k$. With this set up, the finite buffer restriction can also be seen as the state from which customer balks with probability 1(=k/k). There is no balking from an empty queue.

For reneging, we shall assume that customers joining the system are of Markovian reneging type. We shall assume that on joining the system, the customer is aware of its position in the same. Consequently, the reneging rate is taken as a function of the customer's position in the system. In particular, a customer who is at state 'n' will be assumed to have random patience time following exp (ν_n). Under R_BOS, we shall assume that

$$v_n = \begin{cases} 0 & \text{, for } n = 0, 1. \\ v + c^{n-1} & \text{, for } n = 2, 3, \ldots, k \end{cases}$$

and under R_EOS,

$$v_n = \begin{cases} 0 & \text{for } n = 0. \\ v + c^{n-1} & \text{for } n = 1, 2, \ldots, k \end{cases}$$

where c is a constant (c$\geq$0 and c$\neq$1).

Our aim behind this formulation is to ensure that customer's at higher positions have monotonic reneging rates. This requires that as a customer progresses in the system from position n to (n-1), the reneging distribution shift from $\exp(v_n)$ to $\exp(v_{n-1})$. In view of the memory less property, this shifting of reneging distribution is mathematically tractable as we shall demonstrate in the subsequent sections.

Our work stands out on a number of counts. First, one can observe from section 2 that existing reneging literature does not analyze the case where the reneging behavior is position dependent. All such Markovian reneging rules assume that reneging rate is constant irrespective of the position of the customer. To the best of our knowledge, formulation of position dependent reneging rule has not been attempted in literature. However, as mentioned earlier, there are many systems where customers are position aware and hence have variable reneging rates. This formulation is an important focus of this paper. Second, even though one can observe reneging and balking in our day-to-day life, very little work has analyzed these features together. This has been attempted here. Third, an important focus of this paper is the derivation of explicit closed form expressions of performance measures which can be used 'off the shelf' by practitioners. Reneging and balking literature seldom provide explicit closed form expressions; much less when reneging and balking are both involved as in the case of this paper.

The steady state probabilities are derived by the Markov process method. We first analyze the case where customers renege only from the queue. Under R_BOS, letp_n denote the probability that there are 'n' customers in the system. The steady state probabilities under R_BOS are given below,

$$\lambda p_0 = \mu p_1 \tag{3.1}$$

$$\begin{aligned}\lambda\{1-(n-1)/k\}p_{n-1} + \{\mu + nv + c(c^n-1)/(c-1)\}p_{n+1} = \lambda\{1-n/k\}p_n + \\ \{\mu + (n-1)v + c(c^n-1)/(c-1)\}p_{n-1}\ ;\ n = 1, 2, \ldots, k-1\end{aligned} \tag{3.2}$$

$$\lambda\{1-(k-1)/k\}p_{k-1}=\left\{\mu+(k-1)v+c(c^{k-1}-1)/(c-1)\right\}p_k \tag{3.3}$$

Solving recursively, we get (under R_BOS)

$$p_n=\left[\left\{\lambda^n\prod_{r=1}^{k}(1-\overline{r-1}/k\right\}/\left\{\prod_{r=1}^{n}\left(\mu+\overline{r-1}v+c\overline{c^{r-1}-1}/\overline{c-1}\right)\right\}\right]p_0\ ;\ n=1,2,\ldots,k \tag{3.4}$$

where p_0 is obtained from the normalizing condition $\sum_{n=0}^{k}p_n=1$ and is given as

$$p_0=\left[1+\sum_{n=1}^{k}\left\{\lambda^n\prod_{r=1}^{k}(1-\overline{r-1}/k\right\}/\left\{\prod_{r=1}^{n}\left(\mu+\overline{r-1}v+c\overline{c^{r-1}-1}/\overline{c-1}\right)\right\}\right]^{-1} \tag{3.5}$$

The steady state probabilities satisfy the recurrence relation, under R_BOS

$$p_n=\left[\{\lambda(1-\overline{n-1}/k)\}/\{\mu+\overline{n-1}\,\nu+c(c^{n-1}-1)/(c-1)\}\right]p_{n-1};n=1,2,\ldots,k$$

Under R_EOS where customers may renege from queue as well as while being served, let q_n denote the probability that there are n customers in the system. Applying the Markov theory, we obtain the following set of steady state equations.

$$\lambda q_0=(\mu+v)q_1 \tag{3.6}$$

$$\begin{aligned}\lambda\{1-(n-1)/k\}q_{n-1}+\{\mu+(n+1)v+c(c^n-1)/(c-1)\}q_{n+1}&=\lambda\{1-n/k\}q_n\\ +\left\{\mu+nv+c(c^{n-1}-1)/(c-1)\right\}q_n\ &;\ n=1,2,\ldots,k-1\end{aligned} \tag{3.7}$$

$$\lambda\{1-(k-1)/k\}q_{k-1}=\left\{\mu+kv+c(c^{k-1}-1)/(c-1)\right\}q_k \tag{3.8}$$

Solving recursively, we get (under R_EOS)

$$q_n=\left[\left\{\lambda^n\prod_{r=1}^{n}(1-\overline{r-1}/k\right\}/\left\{\prod_{r=1}^{n}\left(\mu+rv+c\overline{c^{r-1}-1}/\overline{c-1}\right)\right\}\right]q_0\ ;\ n=1,2,\ldots,k \tag{3.9}$$

where q_0 is obtained from the normalizing condition $\sum_{n=0}^{k}q_n=1$ and is given as

$$q_0=\left[1+\sum_{n=1}^{k}\left\{\lambda^n\prod_{r=1}^{k}(1-\overline{r-1}/k\right\}/\left\{\prod_{r=1}^{n}\left(\mu+rv+c\overline{c^{r-1}-1}/\overline{c-1}\right)\right\}\right]^{-1} \tag{3.10}$$

The steady state probabilities satisfy the recurrence relation, under R_EOS

$$q_n=\left[\left\{\lambda(1-\overline{n-1}/k)\right\}/\left\{\mu+nv+c(c^{n-1}-1)/(c-1)\right\}\right]q_{n-1}\ ;\ n=1,2,\ldots,k$$

4. Performance Measures

The main objective of any queuing study is to assess some well-defined parameters through which the nature of the quality of service can be studied. These parameters are known as performance measures. Performance measures are important as issues or problems caused by queuing situations are often related to customer's dissatisfaction with service or may be the root cause of economic losses in a business. Analysis of the relevant performance measures of queuing models allows the cause of queuing issues to be identified and the impact of proposed changes to be assessed.

An important measure is the mean number of customers in the system, which is traditionally denoted by 'L'. We have presented the derivation of this important performance measure separately for the two reneging disciplines in the appendix. These are denoted by $L_{\text{R\_BOS}}$ and $L_{\text{R\_EOS}}$.

Let $P(s)$ be the p.g.f of the steady state probability under R_BOS rule. Then we note that

$$\begin{aligned} L_{\text{R\_BOS}} &= \sum_{n=0}^{k} np_n \\ &= P'(1) \\ &= \frac{d}{ds}P(s)\,|_{s=1} \end{aligned}$$

(See the appendix for more derivations)

From (A.8) and (B.2), the mean system sizes under the two reneging rules are

$$L_{\text{R\_BOS}} = k[\lambda-(\mu-\nu)(1-p_0)-p_0+\{c-(p_0/p_0(c\lambda,\mu,\nu,k))/(c-1)\}]/(\lambda+k\nu) \tag{4.1}$$

$$L_{\text{R\_EOS}} = k[\lambda-\mu(1-q_0)-q_0+\{c-(q_0/q_0(c\lambda,\mu,\nu,k))/(c-1)\}]/(\lambda+k\nu) \tag{4.2}$$

The mean queue size formulas for the two cases can now be obtained and are given by

$$\begin{aligned} L_{q(\text{R\_BOS})} &= \sum_{n=2}^{k}(n-1)p_n \\ &= L_{R\_BOS}-(1-p_0) \\ &= k[\lambda-(\mu-\nu)(1-p_0)-p_0+\{c-(p_0/p_0(c\lambda,\mu,\nu,k))/(c-1)\}]/(\lambda+k\nu)-(1-p_0) \end{aligned}$$

where p_0 and $p_0(c\lambda,\mu,\nu,k)$ are defined in (3.5) and (A.4) respectively.
Similarly,

$$\begin{aligned} L_{q(R\_EOS)} &= \sum_{n=2}^{k}(n-1)q_n \\ &= L_{R\_EOS}-(1-q_0) \\ &= k[\lambda-\mu(1-q_0)-q_0+\{c-(q_0/q_0(c\lambda,\mu,\nu,k))/(c-1)\}]/(\lambda+k\nu)-(1-q_0) \end{aligned}$$

where q_0 and $q_0(c\lambda,\mu,\nu,k)$ are defined in (3.10) and (B.3) respectively.

Customers arrive into the system at the rate λ. However all the customers who arrive do not join the system because of balking and due to finite buffer restriction. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\begin{aligned} {\lambda^e}_{(\text{R\_BOS})} &= \lambda\sum_{n=0}^{k}(1-n/k)p_n \\ &= \lambda[1-L_{\text{R\_BOS}}/k] \\ &= \lambda[k\nu+(\mu-\nu)(1-p_0)+p_0-\{c-(p_0/p_0(c\lambda,\mu,\nu,k))/(c-1)\}]/(\lambda+k\nu) \end{aligned} \tag{4.3}$$

Similarly in case of R_EOS

$$\lambda^{\mathrm{e}}{}_{R\_EOS} = \lambda[k\nu + \mu(1-q_0) + q_0 - \{c - (q_0/q_0(c\lambda,\mu,\nu,k))/(c-1)\}]/(\lambda + k\nu) \tag{4.4}$$

We have assumed that each customer has a random patience time following $\exp(v)$. Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate (avg rr) under two reneging rules are given by

$$\begin{aligned} Avgrr_{(R\_\mathrm{BOS})} &= \sum_{n=2}^{k}\{(n-1)\nu + c(c^{n-1}-1)/(c-1)\}q_n \\ &= \nu\{P'(1) - p_1\} - \nu(1-p_0-p_1) + \{1\big/(c-1)\}\sum_{n=2}^{k} c^n p_n - \{c/(c-1)\}\sum_{n=2}^{k} p_n \\ &= \nu L_{R\_BOS} - \nu(1-p_0) - c/(c-1) + p_0/\{p_0(c\lambda,\mu,\nu,k)(c-1)\} + p_0 \end{aligned} \tag{4.5}$$

$$\begin{aligned} Avgrr_{(R\_\mathrm{EOS})} &= \sum_{n=1}^{k}\{n\nu + c(c^{n-1}-1)/(c-1)\}q_n \\ &= \nu Q'(1) + \{1\big/(c-1)\}\sum_{n=1}^{k} c^n p_n - \{c/(c-1)\}\sum_{n=1}^{k} q_n \\ &= \nu L_{R\_EOS} + c/(c-1) - q_0/\{q_0(c\lambda,\mu,\nu,k)(c-1)\} + q_0 \end{aligned} \tag{4.6}$$

In a real life situation, customers who balk or renege represent the business lost. Customers are lost to the system in three ways, due to balking, due to finite buffer restriction and due to reneging. Management would like to know the proportion of total customers lost in order to have an idea of total business lost.

Hence the mean rate at which customers are lost (under R_BOS) is

$$\lambda - \lambda^e{}_{(R\_BOS)} + avgrr_{(R\_BOS)} = \lambda - \mu(1-p_0) \tag{4.7}$$

and the mean rate at which customers are lost (under R_EOS) is

$$\lambda - \lambda^e{}_{(R\_EOS)} + avgrr_{(R\_EOS)} = \lambda - \mu(1-q_0) \tag{4.8}$$

These rates helps in the determination of proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion (under R_BOS) is given by

$$\lambda - \lambda^e{}_{(R\_BOS)} + avgrr_{(R\_BOS)}\big/\lambda = 1 - (\mu/\lambda)(1-p_0)$$

and the proportion (under R_EOS) is given by

$$\lambda - \lambda^e{}_{(R\_EOS)} + avgrr_{(R\_EOS)}\big/\lambda = 1 - (\mu/\lambda)(1-q_0)$$

The proportion of customer completing receipt of service can now be easily determined from the above proportion.

The customers who leave the system from the queue do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work he has to do. Let us call the rate at which customers reach the service station as λ^s .Then under R_BOS

$$\begin{aligned} &= \lambda^e{}_{(R\_BOS)}\left\{1 - \sum_{n=2}^{k}(n-1)\nu p_n/\lambda^e{}_{(R\_BOS)}\right\} \\ &= \lambda^e{}_{(R\_BOS)} - avgrr_{(R\_BOS)} \\ &= \mu(1-p_0) \end{aligned}$$

In case of R_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus,

$$\begin{aligned}
&\lambda^{s}_{(\text{R_EOS})} = \lambda^{e}_{(\text{R_EOS})}(\text{1-proportion of customers lost due to reneging out of those joining the system})\\
&= \lambda^{e}{}_{(\text{R_EOS})}\left\{1 - \sum_{n=2}^{k}(n-1)\nu q_n / \lambda^{e}{}_{(\text{R_EOS})}\right\}\\
&= \lambda^{e}{}_{(R_EOS)} - avgrr_{(R_EOS)}\\
&= \mu(1-q_0)
\end{aligned}$$

5. Sensitivity Analysis

We have assumed that there are essentially four parameters viz: λ, μ, ν, k relating to the stochastic nature of arrival, service, reneging patterns and system size. Various reasons may influence these parameters so that on different occasions these may undergo change. From managerial point of view, an idle server is a waste. So also for low server utilization. It is therefore interesting to examine and understand how server utilization varies in response to change in system parameters. We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section.

$p_n\ (\lambda, \mu, \nu, k)$ and $q_n\ (\lambda, \mu, \nu, k)$ will denote the probability that there are 'n' customers in a system with parameters in (λ, μ, ν, k) stead state under R_BOS and R_EOS respectively.

i. Let $\lambda_1 > \lambda_0$, then

$$\begin{aligned}
&\tfrac{p_0(\lambda_1,\mu,\nu,k)}{p_0(\lambda_0,\mu,\nu,k)} < 1\\
&\Rightarrow \tfrac{(\lambda_0-\lambda_1)}{\mu} + \tfrac{(1-1/k)(\lambda_0{}^2-\lambda_1{}^2)}{\mu(\mu+\nu+c)} + \ldots + \tfrac{(1-1/k)\ldots\{1-(k-1)/k\}(\lambda_0{}^k-\lambda_1{}^k)}{\mu(\mu+\nu+c)\ldots\{\mu+(k-1)\nu+c(c^{k-1}-1)/(c-1)\}} < 0
\end{aligned}$$

which is true. Hence $p_0 \downarrow$ as $\lambda \uparrow$.

ii. Let $\mu_1 > \mu_0$, then

$$\begin{aligned}
&\tfrac{p_0(\lambda,\mu_1,\nu,k)}{p_0(\lambda,\mu_0,\nu,k)} > 1\\
&\Rightarrow \lambda\left(\tfrac{1}{\mu_0} - \tfrac{1}{\mu_1}\right) + \lambda^2(1-1/k)..\left\{\tfrac{1}{\mu_0(\mu_0+\nu+c)} - \tfrac{1}{\mu_1(\mu_1+\nu+c)}\right\} + \ldots +\\
&+\lambda^k(1-1/k)(1-2/k)\ldots\{1-(k-1)/k\}\left\{\begin{array}{c}\frac{1}{\mu_0(\mu_0+\nu+c)\ldots\{\mu_0+(k-1)\nu+c(c^{k-1}-1)/(c-1)\}} - \\ \frac{1}{\mu_1(\mu_1+\nu+c)\ldots\{\mu_1+(k-1)\nu+c(c^{k-1}-1)/(c-1)\}}\end{array}\right\} > 0
\end{aligned}$$

which is true. Hence $p_0 \uparrow$ as $\lambda \uparrow$.

iii. Let $v_1 > v_0$, then

$$\begin{aligned}
&\tfrac{p_0(\lambda,\mu,\nu_1,k)}{p_0(\lambda,\mu,\nu_0,k)} > 1\\
&\Rightarrow \lambda^2(1-1/k)\left\{\tfrac{1}{\mu(\mu+\nu_0+c)} - \tfrac{1}{\mu(\mu+\nu_1+c)}\right\} + \ldots\\
&+\lambda^k(1-1/k)(1-2/k)\ldots.\{1-(k-1)/k\}\left\{\begin{array}{c}\frac{1}{\mu(\mu+\nu_0+c)\ldots\{\mu+(k-1)\nu_0+c(c^{k-1}-1)/(c-1)\}} - \\ \frac{1}{\mu(\mu+\nu_1+c)\ldots\{\mu+(k-1)\nu_1+c(c^{k-1}-1)/(c-1)\}}\end{array}\right\} > 0
\end{aligned}$$

which is true. Hence $p_0 \uparrow$ as $v \uparrow$.

iv. Let $k_1 > k_0$, then

$$\begin{aligned}
&\tfrac{p_0(\lambda,\mu,\nu,k_1)}{p_0(\lambda,\mu,\nu,k_0)} < 1\\
&\Rightarrow \sum_{n=1}^{k_0} \frac{\lambda^n \prod_{r=1}^{n}\{1-(r-1)/k_0\}}{\prod_{r=1}^{n}\left\{\mu+(r-1)\nu+c(c^{r-1}-1)/(c-1)\right\}} - \sum_{n=1}^{k_1} \frac{\lambda^n \prod_{r=1}^{n}\{1-(r-1)/k_1\}}{\prod_{r=1}^{n}\left\{\mu+(r-1)\nu+c(c^{r-1}-1)/(c-1)\right\}} < 0
\end{aligned}$$

which is true. Hence $p_0 \downarrow$ as $k \uparrow$.

The following can similarly be shown
v. $q_0 \downarrow$ as $\lambda \uparrow$
vi. $q_0 \uparrow$ as $\mu \uparrow$
vii. $q_0 \uparrow$ as $v \uparrow$
viii. $q_0 \downarrow$ as $k \uparrow$
Under R_BOS, these results state that an increase in arrival rate would result in lowering of the fraction of time the server is idle. An increase in service rate would mean the server is able to work efficiently so that it can process same amount of work quickly. This translates to higher server idle time. An increase in reneging rate would mean the server has fewer work to do and hence higher fraction of idle time. An increase in system size translates to the lowering of the fraction of time the server is idle. Similar conclusions can be drawn under R_EOS.

6. Numerical Example

To illustrate the use of our results, we apply them to a queuing scenario. We quote below an example from Taha (2003, page 610).

'The time for barber Joe to give a haircut is exponential with mean of 12 minutes. Because of his popularity, customers usually arrive (according to a Poisson distribution) at a rate much higher than Joe can handle 6 customers per hour. Joe really will feel comfortable if the arrival rate is effectively reduced to about 4 customers per hour. To accomplish this goal, he came up with the idea of providing limited seating in the waiting area so that newly arriving customers would go elsewhere when they discover that all the seats are taken. How many seats should Joe provide to accomplish his goal?'

This is a design problem where the system manager (Joe, the barber) desires a system design in respect of size of the waiting area (number of chairs for waiting customers).

Here $\lambda = 6/hr$ and $\mu = 5/hr$. As required by Joe, we examine the effect of limited seating arrangement in the waiting area with different choices of k. Though not explicitly stated, it is necessary to assume reneging and balking. Customers these days are very hard pressed for time. Prompt customer service being the expectation, it is all the more reasonable to assume that customers are all of reneging type. We shall assume that reneging distribution is state dependent following $\exp(v_n)$ where v_n is as described in section 3. Specifically, we shall assume $v = 0.1/hr$ and considered the scenario with $c = 1.05$. Given the fact that service in a barbershop is being analyzed, clearly the reneging rule would be R_BOS. We further assume that the probability of balking by an arriving customer is 'i/k', $i = 1, 2, \ldots, k$ where i is the state of the customer observes the system to be in on its arrival.

Various performance measures of interest computed under different scenarios are given in Table in below. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of k were considered. Results relevant with regard to Joe's desire to limit arrival rate of customers into his service station to something around $4/hr$ are presented in the table.

Since a larger waiting area would also entail additional expenditure/ investment, Joe needs to examine how the performance measures differ across different choices of k. In case the reneging behavior of customer follows exp(0.1) distribution and when $c = 1.05$, it appears from the above table that an ideal choice of k could be 20 (seating space in waiting area =19) with $\lambda^s = 3.99628$.

Two interesting observations can be made from the above table. To a layman, Joe's aim of reducing λ from 6 to 4 effectively boils down to turning away one third of his customers. Our analysis confirms the same. In the scenario examined above, the percentage of customers lost due to reneging together with finite buffer at the level of ideal choice of k hovers very close to one third at 33.39%. Second, at the level of λ^s nearest to 4, the fraction of time Joe would be idle (p_0) is almost constant at 20% in the above scenarios. This results stand to reason.

TABLE 1. Performance measures assuming $\lambda = 6/hr$, $\mu = 5/hr$, $v = 0.1/hr$ and $c = 1.05$

Performance Measure	Size of Waiting Area		
	19 ($k=20$)	20 ($k=21$)	21 ($k=22$)
λ^s (i.e. arrival rate of customers reaching service station)	3.99628	4.00249	4.00816
Effective arrival rate (λ^e)	5.40723	5.43109	5.45309
Fraction of time server is idle	0.20074	0.19950	0.19837
Average length of queue	1.17665	1.19069	1.20369
Average length of system	1.97590	1.99119	2.00532
Mean reneging rate	1.41095	1.42860	1.44494
Rate of loss due to balking and finite buffer	0.59277	0.56891	0.54691
Mean rate of customers lost	2.00371	1.99751	1.99184
Proportion of customers lost due to reneging, balking and finite buffer	0.33395	0.33292	0.33197

7. Conclusion

The analysis of a single server Markovian queuing system with state dependent balking and position dependent reneging has been presented. Even though balking and reneging have been discussed by others, explicit expressions are not always available. Besides, to the best of our knowledge, modeling of position dependent reneging has not been attempted in literature. This paper makes a contribution here. Closed form expressions of number of performance measures have been derived. To study the change in the system corresponding to change in system parameters, sensitivity analysis has also been presented. A numerical example has been discussed to demonstrate results derived. The numerical example is of indicative nature meant to illustrate the benefits of our theoretical results in a design context. The limitations of this work stem from the Markovian assumptions. Extension of our results for general distribution is a pointer to future research.

Appendix

A. Derivation of $P'(1)$ under R_BOS

Let $P(s)$ denote the probability generating function, defined by $P(s) = \sum_{n=0}^{\infty} p_n s^n$. From (3.2) we have

$$\lambda\{1-(n-1)/k\}p_{n-1} + \{\mu + nv + c(c^n-1)/(c-1)\}p_{n+1} = \lambda\{1-n/k\}p_n + \{\mu + (n-1)v + c(c^{n-1}-1)/(c-1)\}p_n \; ; \; n = 1, 2, \ldots, k-1$$

Multiplying both sides of the equation by s^n and summing over n

$$\lambda s \sum_{n=1}^{k-1}\{1-(n-1)/k\}p_{n-1}s^{n-1} - \lambda\sum_{n=1}^{k-1}(1-n/k)p_n s^n = \sum_{n=1}^{k-1}\{\mu + (n-1)v + (c^n - c)/(c-1)\}p_n s^n - \frac{1}{s}\sum_{n=1}^{k-1}\{\mu + nv + (c^{n+1}-c)/(c-1)\}p_{n+1}s^{n+1} \tag{A.1}$$

$\Rightarrow \lambda s[p_0 s^0 + (1-1/k)p_1 s^1 + \ldots + \{1-(k-1)/k\}p_{k-2}s^{k-2}] - \lambda[(1-1/k)p_1 s^1 + (1-2/k)p_2 s^2 + \ldots + \{1-(k-1)/k\}p_{k-1}s^{k-1} = \mu(p_1 s^1 + \ldots + p_{k-1}s^{k-1}) + \nu\{p_2 s^2 + 2p_3 s^3 + \ldots + (k-1)p_k s^k + \{1/(c-1)\}\{cp_1 s + c^2 p_2 s^2 + \ldots + c^{k-1}p_{k-1}s^{k-1} - c(p_1 s^1 + \ldots + p_{k-1}s^{k-1}\} - (1/s)[\mu(p_2 s^2 + \ldots + p_k s^k) + \nu\{p_2 s^2 + 2p_3 s^3 + \ldots + (k-1)p_k s^k + \{1/(c-1)\}\{c^2 p_2 s^2 + \ldots + c^k p_k s^k - c(p_2 s^2 + \ldots + p_k s^k\}$

$\Rightarrow \lambda s[(p_0 s^0 + \ldots + p_{k-2}s^{k-2}) - (1/k)\{p_1 s + 2p_2 s^2 + \ldots + (k-2)p_{k-2}s^{k-2}\}] - \lambda[(p_1 s^1 + \ldots + p_{k-1}s^{k-1}) - (1/k)\{p_1 s + 2p_2 s^2 + \ldots + (k-1)p_{k-1}s^{k-1}\}] = \mu\{P(s) - p_0 - p_k s^k\} + \nu s\{2p_2 s + \ldots + (k-1)p_{k-1}s^{k-2}\} -$

$\nu s(p_2 s + ... + p_{k-1}s^{k-2}) + \{1/(c-1)\}\{P(cs) - p_0 - c^k p_k s^k - cP(s) + cp_0 + cp_k s^k\} - (1/s)[\mu\{P(s) - p_0 - p_1 s\} + \nu s\{2p_2 s + ... + kp_k s^{k-1}\} - \nu s(p_2 s + ... + p_k s^{k-1}) + \{1/(c-1)\}\{P(cs) - p_0 - cP(s) + cp_0\}]$

$\Rightarrow \lambda s[\{P(s) - p_{k-1}s^{k-1} - p_k s^k\} - (s/k)\{P'(s) - (k-1)p_{k-1}s^{k-2} - kp_k s^{k-1}\}] - \lambda[\{P(s) - p_k s^k - p_0\} - (s/k)\{P'(s) - kp_k s^{k-1}\}] = \mu\{P(s) - p_0 - p_k s^k\} + \nu s\{P'(s) - p_1 - kp_k s^{k-1}\} - \nu\{P(s) - p_0 - p_1 s - p_k s^k\} + \{1/(c-1)\}\{P(cs) + p_0(c-1) - cP(s) - cp_k s^k(c^{k-1} - 1)\} - (\mu/s)\{P(s) - p_0 - p_1 s\} - \nu\{P'(s) - p_1\} + (\nu/s)\{P(s) - p_0 - p_1 s\} - \{1/(c-1)s\}\{P(cs) + p_0(c-1) - cP(s)$

$\Rightarrow \{(\lambda s/k) + \nu\}P'(s)(1-s) = (\nu/s)P(s)(1-s) - (\mu/s)P(s)(1-s) + (\mu/s)p_0(1-s) - (\nu/s)p_0(1-s) - \{P(cs)(1-s)/(c-1)s\} - (p_0/s)(1-s) + \lambda P(s)(1-s) + \{\mu + (k-1) + c(c^{k-1}-1)/(c-1)\}p_k s^k - \{\mu - \nu + \nu k + c(c^{k-1}-1)/(c-1)\}p_k s^k$

$\Rightarrow \{(\lambda s + k\nu)/k\}P'(s) = [(\nu/s) - (\mu/s) + \lambda + c/\{s(c-1)\}]P(s) + \{(\mu/s) - (\nu/s) - 1/s\}p_0 - P(cs)/\{(c-1)s\}$

Now

$\lim_{s\to 1^-} P'(s) = \lim_{s\to 1^-} \{k/(\lambda + k\nu)\}[\{(\nu/s) - (\mu/s) + \lambda + c/(c-1)s\}P(s) + \{(\mu/s) - (\nu/s) - 1/s)p_0 - \{P(c)/(c-1)s\}$

$$\Rightarrow P'(1) = \{k/(\lambda + k\nu)\}[\lambda - (\mu - \nu)(1 - p_0) - p_0 + \{c - P(c)\}/(c-1)] \tag{A.2}$$

Here $P(c) = \sum_{n=0}^{k} p_n(\lambda,\mu,\nu,k)c^n$ where the symbol $p_n(\lambda,\mu,\nu,k)$ is as described in section 5. We use p_n and $p_n(\lambda,\mu,\nu,k)$ interchangeably. However should any of the parameters λ,μ,v,k change, it is explicitly stated. To obtain a closed form expression for $P(c)$, let us for the time being, consider another queuing system with parameter and assumptions similar to the queuing system we are presently considering except that the arrival rate is '$c\lambda$'. For this new system, the steady state equations are same as (3.1), (3.2) and (3.3) with 'λ' replaced by '$c\lambda$'. Denoting the steady state probabilities of this new system by $p_n(c\lambda,\mu,\nu,k)$, we can obtain

$$p_n(c\lambda,\mu,\nu,k) = \left[(c\lambda)^n \prod_{r=1}^{n}\{1-(r-1)/k\} \Big/ \left\{\prod_{r=1}^{n}\left(\mu + \overline{r-1}v + c\overline{c^{r-1}-1}/\overline{c-1}\right)\right\}\right] p_0(c\lambda,\mu,\nu,k)$$

$$; n = 1,2,\ldots,k \tag{A.3}$$

where

$$p_0(c\lambda,\mu,\nu,k) = \left[1 + \sum_{n=1}^{k}(c\lambda)^n \prod_{r=1}^{n}\{1-(r-1)/k\} \Big/ \left\{\prod_{r=1}^{n}\left(\mu + \overline{r-1}v + c\overline{c^{r-1}-1}/\overline{c-1}\right)\right\}\right]^{-1} \tag{A.4}$$

Let $P(S;c\lambda,\mu,\nu,k)$ denotes the probability generating function of this new queuing system so that

$$P(S;c\lambda,\mu,\nu,k) = \sum_{n=0}^{k} p_n(c\lambda,\mu,\nu,k)s^n$$

Now

$$\begin{aligned} P(c) &= \sum_{n=0}^{k} p_n(\lambda,\mu,\nu,k)c^n \\ &= p_0 + \sum_{n=1}^{k}\left[(c\lambda)^n \prod_{r=1}^{n}\{1-(r-1)/k\} \Big/ \{\prod_{r=1}^{n}(\mu + \overline{r-1}\,\nu + c\overline{c^{r-1}-1}/\overline{c-1})\}\right] p_0 \\ \Rightarrow [\{P(c) - p_0\}/p_0] &= \sum_{n=1}^{k}\left[(c\lambda)^n \prod_{r=1}^{n}\{1-(r-1)/k\} \Big/ \{\prod_{r=1}^{n}(\mu + \overline{r-1}\,\nu + c\overline{c^{r-1}-1}/\overline{c-1})\}\right] p_0 \end{aligned} \tag{A.5}$$

Now putting $S = 1$ in $P(S; c\lambda, \mu, \nu, k)$ we get

$$\begin{aligned}
&P(1; c\lambda, \mu, \nu, k) = p_0(c\lambda, \mu, \nu, k) + \sum_{n=1}^{k} p_n(c\lambda, \mu, \nu, k) \\
&\Rightarrow 1 = p_0(c\lambda, \mu, \nu, k) + \sum_{n=1}^{k} \left[(c\lambda)^n \prod_{r=1}^{n} \{1 - (r-1)/k\} \Big/ \{ \prod_{r=1}^{n} (\mu + \overline{r-1}\,\nu + c\overline{c^{r-1} - 1}/\overline{c-1}) \} \right] p_0(c\lambda, \mu, \nu, k) \\
&\Rightarrow 1 = p_0(c\lambda, \mu, \nu, k) + \{(P(c) - p_0)/p_0\}\, p_0(c\lambda,\mu,\nu,k) \\
&\Rightarrow P(c) = p_0/p_0(c\lambda, \mu, \nu, k)
\end{aligned} \tag{A.6}$$

{using (A.3) and (A.5)}

Similarly under R_EOS,

$$Q(c) = q_0/q_0(c\lambda, \mu, \nu, k) \tag{A.7}$$

Using equation (A.6) in (A.2) we obtain

$$P'(1) = \{k/(\lambda + k\nu)\}[\lambda - (\mu - \nu)(1 - p_0) - p_0 + \{c - p_0/p_0(c\lambda, \mu, \nu, k)\}/(c-1)] \tag{A.8}$$

where $p_0(c\lambda, \mu, \nu, k)$ is given in (A.4).

B. Derivation of $Q'(1)$ under R_EOS

From equation (3.7), we have

$$\begin{aligned}
\lambda\{1 - (n-1)/k\}q_{n-1} + \{\mu + (n+1)\nu + c(c^n - 1)/(c-1)\}\, q_{n+1} &= \lambda(1 - n/k)q_n \\
&+ \left\{\mu + n\nu + c(c^{n-1} - 1)/(c-1)\right\} q_n; n = 1, 2, \ldots, k-1
\end{aligned}$$

Multiplying both sides of this equation by s^n and summing over n we get

$$\begin{aligned}
\lambda s \sum_{n=1}^{k-1} \{1 - (n-1)/k\} q_{n-1} s^{n-1} - \lambda \sum_{n=1}^{k-1} \{1 - n/k\} q_n s^n &= \sum_{n=1}^{k-1} \{\mu + n\nu + (c^n - c)/(c-1)\} q_n s^n \\
&- \frac{1}{s} \sum_{n=1}^{k-1} \{\mu + (n+1)\nu + (c^{n+1} - c)/(c-1)\} q_{n+1} s^{n+1}
\end{aligned} \tag{B.1}$$

Proceeding in a manner similar to previous section, we obtain

$$Q'(1) = \{k/(\lambda + k\nu)\}[\lambda - \mu(1 - q_0) - q_0 + \{c - q_0/q_0(c\lambda, \mu, \nu, k)\}/(c-1)] \tag{B.2}$$

where

$$q_0(c\lambda, \mu, \nu, k) = \left[1 + \sum_{n=1}^{k} (c\lambda)^n \prod_{r=1}^{n} \{1 - (r-1)/k\} \Big/ \{ \prod_{r=1}^{n} (\mu + r\nu + c\overline{c^{r-1} - 1}/\overline{c-1}) \} \right]^{-1} \tag{B.3}$$

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