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ORDERING OF CLAIM SIZE RISKS AND ACTUARIAL IMPLICATIONS

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Abstract : Ordering of claim size risks can be done by stochastic dominance, stop-loss dominance and several other modalities of stochastic orders for decision making under uncertainty. Emergence of risks can be deteected in terms of some interdependent variables and factors such as risk reserves and premium amounts. Risk ordering must be done by models that take account of this interdependence.

This paper discusses some major stochastic order modalities for comparisons of risks and introduces a new risk ordering criterion in terms or record values of claim size sequences. Comparison of two independent portfolios is shown by these risk ordering approaches. .

Key words: Record values, distribution of total claim amount, risk orderings

1. Introduction : Stochastic ordering aims to build consistent methods for comparing selected characteristics of competing phenomena that are of interest for decision making under uncertainty. Certain events can be defined as risk in terms of these characteristics and their probability distributions. Selection of minimum risk events is the objective in decision making. A criterion to assess the minimality of a risk is smallness of that risk in comparison to another one which is defined on the same probability space. In this sense, determination of stochastic orders of risk events have been a subject matter in actuarial applications of risk theory. The ultimate task in these applications is to set valid grounds in order to specify insurance and reinsurance strategies under several conditions.

There are several modes of risk ordering [5] [12]. Some mostly used used ones of these modes are discussed in the following section with a synthesizing approach. Then, the concept of risk ordering by record values is introduced. Stochastic dominance of a claim size sequence of records over another one is discussed. Actuarial decision making implications of this new mode of risk ordering open a broadened perspective for the purposes of selecting less risky insurance portfolios.

2. Stochastic Dominance and Risk Ordering : Let $[t_{j-1}, t_j)$ be equal lenght non-overlapping time periods whose union is equal to $[0, t)$, the total time period for actuarial practice. Also, let N_{t_j} and Y_{t_j} be number of loss occurance and the resulting loss amount claimable in a $[t_{j-1}, t_j)$ time period which are assumed to be independent. N_{t_j} is taken as a random variable with $Poisson\,(\lambda_j)$.

N_t is a $Poisson\,(\lambda)$ distributed random variable with λ as a sum of λ_j's. Given that Y_{t_j}'s are iid sequence of random variables, the risk for $[0, t)$ period is the random total loss amount (aggregate claim)

$$S_t = \sum_{i=1}^{N_t} Y_i \tag{1}$$

whose distribution function is compound poisson distribution

$$F_{S_i}\,(s) = \sum_{n=0}^{\infty} \Pr\,(N_t = n)\, F_Y^{*n}\,(s) \tag{2}$$

where $F_Y^{*n}(s)$ is the n-th convolution of distribution function of Y_j's. If, structure parameter λ is also a random variable, the expression (2) becomes an expression for mixed compound poisson process. Properties of distribution $F_{S_i}(s)$ is known [3] [7].

Mosler and Scarsini [8], and Shaked and Shantikumar [12], among others, give an explicit discussion of stochastic orders and their applications which can be extended easily to ordering of risks. Goovaerts et.all. [5] discuss risk ordering with lucid actuarial implications.

Consider S_{1t} and S_{2t} as two different aggregate claim (risk) random variables conceivable in a $[0, t)$ period. Let s be seen as a value which is of critical importance from the viewpoint of any premium, retention limit, priority or a similar quantity. Let F_1 and F_2 be distribution functions of S_{1t} and S_{2t} respectively. Risk S_{2t} stochastically dominates risk S_{1t} if for a real values non-decreasing function $g(.)$,

$$E[g(S_{1t})] \leq E[g(S_{2t})] \tag{3}$$

which is denoted by ordering $S_{1t} \prec_{st} S_{2t}$. The order relation between the risk under stochastic dominance is shown as

$$F_1(s) \geq F_2(s) \ \ \forall s \text{ if } S_{1t} \prec_{st} S_{2t} \tag{4}$$

Some important risk orderings are expressed below for further discussion.

First Order Stochastic Dominance

$$S_{1t} \prec S_{2t} :\leftrightarrow \int_s^\infty (1 - F_1(z))\, dz \leq \int_s^\infty (1 - F_2(z))\, dz, \quad \forall s \geq 0 \tag{5}$$

Stop-loss Dominance

$$S_{1t} \prec_{sl} S_{2t} :\leftrightarrow \int_s^\infty (z - s)\, F_1(z)\, dz \leq \int_s^\infty (z - s)\, F_2(z)\, dz, \quad \forall s \geq 0 \tag{6}$$

Failing Reserve Against Initial Surplus Dominance

$$S_{1t} \prec_{fr} S_{2t} :\leftrightarrow \frac{1}{E(S_{1t})} \int_s^\infty (1 - F_1(z))\, dz \leq \frac{1}{E(S_{2t})} \int_s^\infty (1 - F_2(z))\, dz, \quad \forall s \geq 0 \tag{7}$$

such that $S_{1t} \prec_{fr} S_{2t}$, $E(S_{1t}) = E(S_{2t}) \rightarrow E(S_{1t}^2) \leq E(S_{2t}^2)$ and $Var(S_{1t}) = Var(S_{2t})$

Dangerous Claim Size Distribution

$$S_{1t} \prec_k S_{2t} :\leftrightarrow \left[1 - \int_0^s (1 - F_1(zE(S_{1t})))\, dz\right] \leq \left[1 - \int_0^s (1 - F_2(zE(S_{2t})))\, dz\right], \quad \forall s \geq 0 \tag{8}$$

Conditional Expected Claim Size Dominance

$$S_{1t} \prec_{cs} S_{2t} :\leftrightarrow E\left[(S_{1t} - s)^+ | S_{1t} > s\right] \leq E\left[(S_{2t} - s)^+ | S_{2t} > s\right], \quad \forall s \geq 0 \tag{9}$$

such that

$$\frac{1}{1 - F_1(s)} \int_s^\infty (1 - F_1(z))\, dz \leq \frac{1}{1 - F_2(s)} \int_s^\infty (1 - F_2(z))\, dz \tag{10}$$

The direct implication of these risk orderings for actuarial decision making is that the pure premium for case S_{1t} is to be less than that of S_{2t}. There is higher risk for the case of S_{2t} regarding the exceedance of total claim size over a limit, hence the reinsurance policies requires more protractive measure against risk S_{2t} than that of S_{2t}.

The rating and comparison of above stochastic order relations may show that they can imply each other [7]. The distribution of S_{1t} and S_{2t} are required to belong to the same class for comparing them. A dominance relation between these two independent total claim size can be reflected through a relation between parameters of their distributions. Note that the most important characteristics of distributions are their moments up to order three at most which are used for determining premiums and many other insurance policies.

3. Record Values of Claim Size Sequences and Risk Ordering : Let $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ be an independent sequence of loss amounts (risk) for a $[0, t)$ time period for two different conceivable claim size situations, $i = 1, 2$. Define two sequence of random variables $U_{ij},\ i = 1, 2,\ j = 1, 2, \ldots, n$ as follows

$$U_{i1} = 1,\, U_{ij} = \min\left\{t : t > U_{ij-1},\, Y_{ij} > Y_{i,U_{ij-1}},\, j > 1\right\} \tag{11}$$

for each i independently. The random variables U_{ij} are called upper record times. The sequence of random variables for $i = 1, 2$ $Y_{iU_1}, Y_{iU_2}, \ldots, Y_{iU_i}$ are called the record values of sequence $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ whose sum equals S_{it}, $i = 1, 2$, the random total loss (claim) amount (size). By this definition, the record values of a sequence of claim sizes constitute an extremal process that produce values larger than previous ones. Record values are important because of asymptotic theory and weak convergence applications. In other words, the succession of states visited by $\left\{Y_{iU_{ij}},\, j \geq 1\right\}$ is record values and they have the embeddings of a Markov process of states visited.

The stochastic dominance of risks S_{1t} and S_{2t} over each other can be discussed from the viewpoint of tails of distributions of S_{it}'s. Other things being equal, the risk with less heavy upper tails is selected. Note that in a sequence of loss amounts, the number of record values can be 1 to n, and this avails to have substantial information for risk assessment by record values. Accordingly the stop-loss risk ordering appeal to be the effective decision criterion for the selection of less risky situations. Hence, we assert that record value distributions for two different portfolios can be taken as strong tools for comparison of risks involved in these portfolios.

One point that enhances this assertion is the equivalence of stop-loss risk ordering and variability risk ordering. We say that S_{1t} is less variable than S_{2t}, written $S_{1t} \prec_v S_{2t}$, if there exist a random variable D such that $S_{1t} + D$ has the same distribution as S_{2t} with the additional property that

$$\Pr\left\{E\left(D|S_{1t}\right) \geq 0\right\} = 1 \tag{12}$$

One can not say that $Var\left(S_{2t}\right) \geq Var\left(S_{1t}\right)$ if S_{2t} is more variable than S_{1t}. Hence the above mentioned equivalence can only be depicted as the following theorem says:

Theorem 1. *$S_{1t} \prec_{sl} S_{2t}$ and $S_{1t} \prec_v S_{2t}$ are equivalent for risks S_{1t} and S_{2t} only in the sense that*

$$S_{1t} \prec_v S_{2t} \rightarrow S_{1t} \prec_{sl} S_{2t} \tag{13}$$

T ake $S_{2t} = S_{1t} + D$ with probability one. Applying Jensen's inequality to the convex non-decreasing function D and to the conditional distribution of $(S_{2t} - (S_{1t} - D))$ given S_{1t} yields

$$E\left[\left(S_{2t} - D\right)^+\right] \geq E_{S_{1t}}\left[\left(S_{1t} + E\left(D|S_{1t}\right) - D\right)^+\right] = E\left[\left(S_{1t} - D\right)^+\right] \tag{14}$$

To prove that (13) is true, it is sufficient to prove the results for discrete random variables and then extend it at limits ■.

Record as an extremal process was discussed and presented a review of record values and related statistics [11] [9]. Most recently, Ahsanullah [1] covered a comprehensive presentation of record statistics and their so far known distributional properties. Along the lines of results that can be found in these works and in the works cited by them, we can say that for risk situation $i=1,2$;

$$\Pr(\mathrm{Y}_{iU_{ij}} > y | \mathrm{Y}_{iU_{ij-1}} = y') = \begin{cases} \frac{1-F_i(y)}{1-F_i(y')} & , y > y' \\ 1 & , y \leq y'. \end{cases} \quad (15)$$

Using this property we can write

$$1 - F_{U_{ij}}(y) = [1 - F_i(y)] \sum_{k=0}^{j} \frac{-\log(1 - F_i(y))}{j!} \tag{16}$$

where $F_{U_{ij}}$ şs the distribution function of record value $Y_{iU_{ij}}$, and $F_i(y)$ the distribution function for claim size Y. Distributional properties (3) and (16) imply that the sequence of record values forms a Markov Chain as said before. Gupta [6] gives details on this matter. Having this in mind, we can say that if the first record value for portfolio i is unbounded, the distribution function of j-th record approximates to

$$F_{U_{ij}} = \frac{[F_i(y)]^{j+1}}{(j+1)!} \text{ as } y \to \infty \tag{17}$$

and its upper tail above y, $1 - F_{U_{ij}}(y)$, to

$$\frac{[1 - F_i(y)]\,[-\log(1 - F_i(y))]^j}{j!} \text{ as } y \to \infty \tag{18}$$

Nagaraya [9] first presented the conditions for existence of expected value of j-th record value of a sequence. $E\left(Y_{iU_{ij}}\right)$ exist for all j if $E\left(|Y_{i1}|^h\right) > 0$, $h > 1$. If the first record value Y_{ij} is unbounded, then expectation of j-th record value exist if $E\left(Y_{i1}(\log Y_{i1})^j\right)$ exist.

Using these results and from (16) we can write the risk orderings in above given five modes in Section 2 at the occurrence of j-th record value, $1 \leq j \leq n$. Other things being equal, the risk with less $1 - F_{U_{ij}}$ value is rationally selected. We sat that this is the risk ordering by the stochastic dominance of j-th record value of risk, and we denote it by $S_{1t} \prec_{rv} S_{2t}$ if S_{2t} is the stochastically dominant risk by record values.

More concisely, letting $\widetilde{S}_{it}$ stand for the sum of record values for portfolio i, the distribution of $\widetilde{S}_{it}$ is

$$F_{\widetilde{S}_{it}} = \sum_{\widetilde{n}=0}^{\infty} P_{\widetilde{n}} F_{u_{ij}}^{*\widetilde{n}} \tag{19}$$

where $\widetilde{n}$ is the number of record values, $P_{\widetilde{n}}$ is the probability of observing $\widetilde{n}$ number of records, and $F_{u_{ij}}^{*\widetilde{n}}$ is the $\widetilde{n}$-th convolution distribution. This distribution here is rather complex as compared to calculation of convolution that should be tackled for expression (2). It is obvious that the (first order) stochastic dominance (5) and the stop-loss dominance (6) (and others, as well) modalities can be directly applied to risk ordering by record values, setting aside the computational details and complications.

The risk ordering of two portfolios can be done merely by record values of the claim size sequences. In this case the above implied methodology suffices to make clear choices among the competing possible portfolios. However, record values are just a subset of a parent sequence of claim sizes and therefore any decision based on risk ordering by record values mode must assume all the other things are equal.

4. $J-$Record Risk Process and Risk Ordering : If we observe over n time units a collection of those observations on $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ which have relative rank j upon being observed, we obviously have more information than if we just have records. Let us call collection of observation which have relative rank j upon being observed a realization of J-Record process. In restricted information situations a J-Record process can be of very inferential use. For different j, the point process embedding of J-records are iid and yields very attractive results[4].

In this respect, we present our critical contribution by a theorem below. Let $N_{im}(j)$ denote the number of loss occurrences in portfolio i, $i = 1, 2$, that come after the j-th record value and are less than the j-th record value.

Theorem 2. *For any m, $j = 1, 2, \ldots$ the probability function of $N_{im}(j)$ is*

$$\Pr\{N_{im}(j) = n'\} = \frac{\binom{m}{n'}}{(j-1)!} \int_a^\infty e^{z\left(m-n'+1\right)} \left(1-e^{-z}\right)^a z^{j-1} dz \tag{20}$$

with

$$E[N_{im}(j)] = m\left(1 - \left(\frac{1}{2}\right)^j\right) \tag{21}$$

and

$$Var[N_{im}(j)] = \left[m^2\left(\left(\frac{1}{3}\right)^j - \left(\frac{1}{2}\right)^{2j}\right)\right] + \left[m\left(\left(\frac{1}{2}\right)^j - \left(\frac{1}{3}\right)^j\right)\right] \tag{22}$$

The limiting distribution of a random variable like $N_{im}(j)$ can be derived as $m \to \infty$. Bairamov [2] presents some results relevant to this case under some regularity conditions ■.

The actuarial implications of Theorem 2 is that when j-th record value occurs at time t' it is possible to order risk $S_{1t'}$ against $S_{2t'}$ and

$$E[N_{1m}(j)]\, E[Y] \le E[N_{2m}(j)]\, E[Y] \tag{23}$$

$$Var[N_{1m}(j)]\, Var[Y] \le Var[N_{2m}(j)]\, Var[Y] \tag{24}$$

by independence and with iid assumption for loss amounts that occur after the j-th record value.

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