PERIODOGRAMS FOR SEASONAL TIME SERIES WITH A UNIT ROOT

Yilmaz Akdi and David A. Dickey
Department of Statistics, Faculty of Science
Ankara University, 06100, Tandogan-Ankara, TURKEY
Department of Statistics, North Carolina State University,
Raleigh, NC 27695, USA
e-mail: akdi@science.ankara.edu.tr

Abstract

This paper extends the periodogram based unit root tests to the seasonal time series models. The problem of testing for a unit root in seasonal time series is something like testing for multiple unit root. That is, in seasonal time series models, there are repeated unit roots. It is found that the asymptotic distribution of the normalized periodogram ordinate is not affected by the seasonality factor and the same test statistic used for autoregressive time series can be used to test for a unit root for seasonal time series models.

1. Introduction

The first priority in seasonal modeling is to specify correct differencing and appropriate transformations. The potential behavior of autocorrelation functions for seasonal models is not easy to characterize. The autocovariance function for a seasonal process is quite complicated. To identify a seasonal model from the sample autocorrelation function of the data, first we find $d$ and $D$ so as to make the differenced observations $X_t = (1-B)^d(1-B^s)^DY_t = \nabla^d\nabla^sDY$ stationary. Next
we examine the sample autocorrelation and partial autocorrelation functions of \( X_t \) at lags which are multiples of \( s \) in order to identify the orders of the model. If \( \hat{\rho}(\cdot) \) is the autocorrelation function of \( X_t \) then the orders \( p \) and \( q \) should be chosen so that \( \hat{\rho}(k) \), \( k = 1, 2, 3, \ldots \), is compatible with the autocorrelation of an \( ARMA(p, q) \) process. The orders \( p \) and \( q \) are then selected by attempting to match \( \hat{\rho}(1), \hat{\rho}(2), \ldots, \hat{\rho}(s-1) \) with the autocorrelation function of an \( ARMA(p, q) \) process. Ultimately the AIC criterion and the goodness of fit test are used to identify the best Seasonal Autoregressive Integrated Moving Average (SARIMA) model among competing alternatives. For autoregressive time series the partial autocorrelation function cuts off after some lags and the autocorrelation function decays exponentially but the rate of the decay is important. And for moving average processes, the partial autocorrelation function decays and the autocorrelation function cuts off after some lags. However, for some time series models, the autocorrelation function may be sinusoidal. Consider the model

\[
y_t - \mu = \rho(y_{t-12} - \mu) + e_t
\]

where \( e_t \) is a sequence of uncorrelated random variables with mean zero and constant variance (white noise). This model is applied to monthly data and expresses this December’s \( y \), for example, as \( \mu \) plus a proportion of last December’s deviation from \( \mu \). If \( \mu = 100 \), \( \rho = 0.8 \), and last December’s \( y \) is 120, the model forecasts this December’s \( y \) as 100+0.8(20)=116. The forecast for next December’s \( y \) is 100+.64(20), and the forecast for \( k \) Decembers ahead is 100+(0.8)^k(20).

The model responds to change in the series because it uses only the most recent December to forecast the future. Suppose we allow \( \rho \) to be 1 in the AR seasonal model. Then the model is nonstationary and reduces to \( y_t = y_{t-12} + e_t \). This model uses last December’s \( y \) as the forecast for next December (and for any other future December). The difference \( y_t - y_{t-12} \) is stationary (white noise). Trend and seasonality are usually detected by inspecting the graph of the (possibly transformed) series. However, they are also characterized by sample autocorrelation functions which are slowly decaying and nearly periodic respectively. Periodograms are usually used to detect periodic components in time series models. Periodograms are also used to estimate the spectral density function.

In this paper we are interested in testing for a unit root in seasonal time series models. Many testing methods have been proposed to test for a unit root in the autoregressive time series models and in the seasonal time series models. The problem arising in many time series applications is the question of whether a series
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should be differenced; this is related to asking if the time series has a unit root. Let \( \{X_t : t = 1, 2, 3, \ldots\} \) be a first order autoregressive process defined by

\[
X_t = \rho X_{t-1} + e_t, \quad X_0 = 0
\]

where \( \{e_t : t = 1, 2, 3, \ldots\} \) is a sequence of independent and identically distributed random variables with \( E(e_t) = 0, \) and \( Var(e_t) = \sigma^2 < \infty. \) Let \( \hat{\rho}_n := \left( \sum X_{t-1}^2 \right)^{-1} \sum X_t X_{t-1} \) be the least squares estimator of \( \rho \) based on the sample of \( n \) observations \( \{X_1, X_2, \ldots, X_n\}. \) The limit distribution of \( \hat{\rho}_n \) is different for the cases: stationary, unstable and explosive. It is normal for the stationary case and nonnormal for the two nonstationary cases. For instance, in the unstable case, \( \rho = 1, \) it is known that

\[
Z_n := \frac{1}{\sigma} \left( \sum_{t=1}^{n} X_{t-1}^2 \right)^{-1/2} (\hat{\rho}_n - 1)
\]

converges weakly to

\[
Z := \frac{1}{2} \left( W^2(1) - 1 \right) \left( \int_0^1 W^2(t) dt \right)^{-1/2}
\]

as \( n \to \infty, \) where \( \{W(t)\} \) is the standard Brownian Motion on \([0, 1].\) Dickey and Fuller (1979) give a representation for the limiting distribution of \( n(\hat{\rho}_n - 1). \) Tables for the percentiles of the distribution can be found in Fuller (1976, pp. 371-3). Dickey, Hasza and Fuller (1984) also give a testing procedure to test for a unit root in seasonal time series. Note that the stationarity of autoregressive time series depends on the roots of the characteristic equation and for seasonal time series, if there is a unit root, then there are more than one unit root. For example, consider the time series model \( y_t = \rho y_{t-2} + e_t \) the corresponding characteristic equation is \( m^2 - \rho = 0. \) If \( \rho = 1 \) then \( m = \pm 1 \) are both roots of the equation. Akdi and Dickey (1998) use a periodogram ordinate to test for a unit root. They derive the exact distribution of the normalized periodogram ordinate for the first order autoregressive time series with a unit root. For the higher order time series, they show that the limiting distribution of the normalized periodogram ordinate remains unchanged. They also give the percentiles of the distribution under the assumption \( \rho = 1. \) In this study, the periodogram based testing procedure has
been extended to seasonal time series with a unit root time series models. It has been shown that limiting distribution remains same for seasonal time series so that same testing procedure developed for first order time series models can be applied to seasonal time series models.

2. The Periodogram Ordinate

The periodogram ordinate is used in many statistical inference problems such as estimating the spectral density function, testing for the presence of a sinusoid with specified frequency, testing for the presence of a Non-Sinusoidal periodic component with specified integer-valued period, and testing for hidden periodicities of unspecified frequency. Spectral analysis for time series, in particular the estimation of the spectral density function, depends heavily on the asymptotic distribution as \( n \to \infty \) of the periodogram ordinates of the series \( \{X_1, X_2, \ldots, X_n\} \). Here, our purpose is to use the periodogram ordinate to test for a unit root in a seasonal time series based on the sample. For stationary time series, there is a one-to-one relationship between the autocorrelation function of the time series and the spectral density function by using the Fourier transformation. However, when \( \rho = 1 \) there is no autocovariance function and hence the spectral density function can not be defined as a Fourier transform of the autocovariance function. But by using the distributional properties derived in this paper, one can define the spectral density function at frequencies near zero.

Periodograms are often used for studying periodic behavior in data. They decompose the variation in data into periodic components. Basic distributional properties of the periodogram ordinates are assumed to be understood. Akedi (1995) studies the distributional properties of the periodogram ordinates for autoregressive time series with a unit root. Akedi and Dickey (1998) give a testing methodology to test for a unit root by using the periodogram ordinates. In this study, a seasonal time series with a unit root satisfying

\[
(Y_t - \mu) = \rho(Y_{t-d} - \mu) + e_t, \quad t = 1, 2, 3, \ldots, n
\]  

(1)

is considered with appropriate starting values and \( e_t \) are assumed to be uncorrelated with mean 0 and variance \( \sigma^2 \). Note that the process is stationary if \(|\rho| < 1\). For such a time series, forecasts tend eventually to the sample mean, and standard estimators of \( \rho \) and \( \mu \) converge to a normal distribution. If \( \rho = 1 \) in (1) and if
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\( Y_0 = \mu \) the forecasts are not mean reverting and the usual estimators of \( \rho \) are nonnormal in the limit. Note that \( \mu \) drops out of (1) if \( \rho = 1 \). Consider the model in (1) with \( \rho = 1 \)

\[ Y_t = Y_{t-d} + e_t, \quad t = 1, 2, 3, \ldots, n \]

The periodogram ordinate of \( Y_t \) is defined as

\[ I_n(w_k) = \frac{n}{2} \left( a_k^2 + b_k^2 \right) \] (2)

where \( a_k \) and \( b_k \) are known as Fourier coefficients of the time series defined by

\[ a_k = \frac{2}{n} \sum_{t=1}^{n} (Y_t - \mu) \cos(w_k t), \quad b_k = \frac{2}{n} \sum_{t=1}^{n} (Y_t - \mu) \sin(w_k t) \] (3)

If \( \mu \) is unknown, then \( \mu \) can be replaced with the sample mean. But when \( w_k = 2\pi k/n, \ k = 1, 2, 3, \ldots [n/2] \) then \( \sum_{t=1}^{n} \cos(w_k t) = \sum_{t=1}^{n} \sin(w_k t) = 0; \) and hence the Fourier coefficients \( a_k \) and \( b_k \) have mean zero when \( Y_t \) has a constant expected value. Here, \([n/2]\) denotes the largest integer less than \( n/2 \). The purpose is to find the distribution of the periodogram ordinate defined in (2) and thus the problem reduces to find the joint distribution of the Fourier coefficients \( a_k \) and \( b_k \) under the assumption \( \rho = 1 \). Note that the periodogram ordinate is a smooth function of a sum of \( e_t \)'s therefore, we can use our results as an approximation as long as \( e_t \) satisfies assumptions of Donsker's theorem and \( n \) is large enough. Therefore, we can assume that errors are independent and normally distributed random variables with mean zero and variance \( \sigma^2 \).

Consider the time series

\[ (Y_t - \mu) = \rho (Y_{t-1} - \mu) + e_t, \quad t = 1, 2, 3, \ldots, n. \]

when \( |\rho| < 1 \) then the normalized periodogram ordinate is asymptotically distributed as chi-square with 2 degrees of freedom. That is,

\[ \frac{I_n(w_k)}{f(w_k)} \overset{D}{\to} \chi^2_2, \quad \text{as} \ n \to \infty \text{ where} \ f(w_k) = \frac{\sigma^2}{1 + \rho^2 - 2\rho \cos(w_k)} \] (4)

and when \( \rho = 1 \), Akdi and Dickey (1998) show that the normalized periodogram ordinate is distributed as mixture of chi-squares with one degree of freedom each. That is, for the first order auto regressive time series

\[ \frac{I_n(w_k)}{f(w_k)} \sim Z_1^2 + 3Z_2^2, \quad \text{where} \ f(w_k) = \frac{\sigma^2}{2(1 - 2 \cos(w_k))} \] (5)

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where \((Z_1, Z_2)\) is a pair of independent standard normals and for a general ARMA(p,q) processes

\[
\frac{4\pi^2 k^2}{\sigma^2 \Phi^2} I_n(w_k) \xrightarrow{D} Z^2_1 + 3Z^2_2 \text{ as } n \to \infty
\]  

(6)

Notice that the low frequency periodogram ordinates are small for stationary time series and large for nonstationary series (variance increases over time for nonstationary case). A test statistic

\[
T_n(w_1) = \frac{2(1 - \cos(w_1))}{\sigma^2_n} I_n(w_1)
\]  

(7)

can be used to test the null hypothesis \(H_0 : \rho = 1\) against stationary alternatives because the distribution of the test statistic is known under both null and alternative hypothesis. The critical values of the null distribution can be found in Akdi and Dickey (1998). Our goal is to show that the same testing procedure can be used to test for a unit root in the seasonal time series with a unit root.

3. Distribution of the Periodogram Ordinate for Seasonal Time Series

Consider the time series given in (1) with \(\rho = 1\). The Fourier coefficients defined in (3) can be written as a sum of independent random variables.

\[
a_k = a_{1,k} + a_{2,k} + \ldots + a_{d,k}, \quad \text{and} \quad b_k = b_{1,k} + b_{2,k} + \ldots + b_{d,k}
\]  

(8)

here

\[
a_{i,k} = \frac{2}{n} \sum_{t=1}^{n} Y_{i,dt-i+1} \cos(w_k(dt - i + 1)) \quad \text{and} \quad b_{i,k} = \frac{2}{n} \sum_{t=1}^{n} Y_{i,dt-i+1} \sin(w_k(dt - i + 1))
\]

such that \(a_{i,k}, a_{j,k}, b_{i,k}, b_{j,k}\) and \(a_{i,k}, b_{j,k}\) are all independent for all \(i \neq j\).

For simplicity let us take \(d = 2\). Then \(Y_t = Y_{t-2} + \epsilon_t\) where \(\epsilon_t\) is a sequence of independent and identically distributed random variables with mean 0 and
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variance $\sigma^2$. $Y_{1,t} = e_1 + e_3 + e_5 + \ldots + e_{d(t-1)}$ and $Y_{2,t} = e_2 + e_4 + e_6 + \ldots + e_{dt}$. In general,

$$Y_{i,t} = \sum_{j=0}^{t-1} e_{i+j} = e_i + e_{i+d} + e_{i+2d} + \ldots + e_{i+d(t-1)}$$

Notice that the random variables $Y_{1,t}$ and $Y_{2,t}$ are independent and thus the Fourier coefficients related to $Y_{1,t}$ and $Y_{2,t}$ are independent. The problem is to find the distribution of the normalized periodogram ordinate which is defined in terms of the Fourier coefficients $a_k$ and $b_k$. Thus we need to find the joint distribution of the Fourier coefficients. Notice that $(a_k, b_k)' = AX$ where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 1 \end{bmatrix}_{2 \times 2d},$$

$$X = (a_{1,k}, b_{1,k}, a_{2,k}, b_{2,k}, \ldots, a_{d,k}, b_{d,k})'$$

In order to find the distribution of $AX$, it is enough to find the joint distribution of $(a_{i,k}, b_{i,k})'$ and since they are normally distributed random variables, we only need to calculate its mean vector and the variance-covariance matrix of $AX$. Obviously, the mean vector is the zero vector. The variances can be calculated as

$$Var(a_{i,k}) = \frac{4\sigma^2}{n^2} \sum_{t=1}^{n} \sum_{s=1}^{n} \min(dt - i + 1, ds - i + 1) \cos \left(2\pi k \frac{dt-i+1}{n}\right) \cos \left(2\pi k \frac{ds-i+1}{n}\right)$$

$$= 4\sigma^2 \sum_{t=1}^{n} \sum_{s=1}^{n} \left\{ \min \left(\frac{dt-i+1}{n}, \frac{ds-i+1}{n}\right) \cos \left(2\pi k \frac{dt-i+1}{n}\right) \cos \left(2\pi k \frac{ds-i+1}{n}\right) \left(\frac{1}{n}\right)^2 \right\}$$

Note that for fixed $k$, this double sum can be approximated as a double integral and after some calculations and using the trigonometric identities we get

$$\frac{1}{n} Var(a_{i,k}) \rightarrow 4\sigma^2 d \int_{0}^{1} \int_{0}^{1} \min(x, y) \cos(2\pi kdx) \cos(2\pi kdy) dx dy = \frac{\sigma^2}{2\pi^2 k^2 d}. $$

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(Here, to avoid notational confusion \( \partial \) is used to indicate the differential). When \( \text{Var}(b_{i,k}) \) and \( \text{Cov}(a_{i,k}, b_{i,k}) \) are calculated similarly

\[
\frac{1}{n} \text{Var}(b_{i,k}) \rightarrow 4\sigma^2 d \int_0^1 \int_0^1 \min(x, y) \sin(2\pi k dx) \sin(2\pi k dy) \partial x \partial y = \frac{3\sigma^2}{2\pi^2 k^2 d}
\]

and

\[
\frac{1}{n} \text{Cov}(a_{i,k}, b_{i,k}) \rightarrow 4\sigma^2 d \int_0^1 \int_0^1 \min(x, y) \cos(2\pi k dx) \sin(2\pi k dy) \partial x \partial y = 0
\]

Thus,

\[
\frac{1}{\sqrt{n}} \begin{bmatrix} a_{i,k} \\ b_{i,k} \end{bmatrix} \xrightarrow{D} N(0, V_1^*) , \text{as } n \to \infty \quad \text{where } V_1^* = \frac{\sigma^2}{2\pi^2 k^2 d} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \tag{9}
\]

But we want to find the distribution of \( (a_k, b_k) = AX \). First of all, the asymptotic distribution of \( X/\sqrt{n} \) is also normally distributed with mean 0 vector and variance covariance matrix \( V_1 \) where \( V_1 = \text{diag}\{V_1^*, V_1^*, \ldots, V_1^*\} \). Thus using the fact that all the components of \( X \) are independent (\( a_{i,k}, a_{j,k}, b_{i,k}, b_{j,k} \) and \( a_{i,k}, b_{j,k} \) independent for all \( i \neq j \)) we can calculate the variance-covariance matrix of \( (a_k, b_k) \). The asymptotic distribution of \( (a_k, b_k) = AX \) is normal with mean vector 0 and variance covariance matrix \( V = AV_1 A' \). The resulting variance-covariance matrix is invariant to the seasonality factor \( d \) and thus the asymptotic distribution is also free from the seasonality factor \( d \).

\[
V = AV_1 A' = \frac{\sigma^2}{2\pi^2 k^2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \tag{10}
\]

Hence,

\[
\frac{1}{\sqrt{n}} \begin{bmatrix} a_k \\ b_k \end{bmatrix} \xrightarrow{D} N(0, V) , \text{as } n \to \infty
\]

This implies that

\[
\frac{1}{\sqrt{n}} (a_k, b_k) \frac{1}{\sqrt{n}} \begin{bmatrix} a_k \\ b_k \end{bmatrix} = \frac{1}{n} (a_k^2 + b_k^2) \xrightarrow{D} \frac{\sigma^2}{2\pi^2 k^2} (Z_1^2 + 3Z_2^2) , \text{as } n \to \infty
\]

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or

\[
\Rightarrow \frac{2\pi^2 k^2}{n \sigma^2} \left( a_k^2 + b_k^2 \right) \xrightarrow{D} Z_1^2 + 3Z_2^2, \text{ as } n \to \infty
\]

\[
\Rightarrow \frac{4\pi^2 k^2 n}{n^2 \sigma^2} \left( a_k^2 + b_k^2 \right) \xrightarrow{D} Z_1^2 + 3Z_2^2, \text{ as } n \to \infty
\]

\[
\Rightarrow \frac{4\pi^2 k^2 n}{n^2 \sigma^2} \left( a_k^2 + b_k^2 \right) \xrightarrow{D} Z_1^2 + 3Z_2^2, \text{ as } n \to \infty
\]

\[
\Rightarrow \left( \frac{2\pi k}{n} \right)^2 \frac{1}{\sigma^2} I_n(w_k) = \frac{w_k^2}{\sigma^2} I_n(w_k) \xrightarrow{D} Z_1^2 + 3Z_2^2, \text{ as } n \to \infty
\]

or

\[
\frac{I_n(w_k)}{f(w_k)} \xrightarrow{D} Z_1^2 + 3Z_2^2, \text{ as } n \to \infty
\]

For fixed \( k \),

\[
\frac{2(1 - \cos(w_k))}{w_k^2} \to 1, \text{ as } n \to \infty
\]

which implies that

\[
\frac{2(1 - \cos(w_k))}{\sigma^2} I_n(w_k) \xrightarrow{D} Z_1^2 + 3Z_2^2, \text{ as } n \to \infty
\]

Therefore we can use the same test statistic given in (7) to test for a unit root in seasonal time series.

4. Example: Testing for Stationarity

In this section, we will try to test the null hypothesis \( H_0 : \rho = 1 \) against stationary alternatives in the model \( y_t = \rho y_{t-4} + e_t \). Note that the value of the periodogram ordinate at the low frequencies is large for time series with unit root and small for stationary time series. And thus, the value of the test statistics is
large for nonstationary time series and small for stationary time series. Therefore, we reject the null hypothesis of unit root if \( T_n(w_k) \) is small. Even though, this testing procedure is valid for any \( k \), it is better to use small \( k \)'s; e.g. \( k = 1 \). The tables for the critical values are available from Akdi and Dickey (1998).

For an illustration we generate 100 observations from a seasonal time series with a unit root: \( y_t = y_{t-4} + e_t \) where \( e_t \) is a sequence of independent normally distributed random variables with mean 0 and variance 1. From identification plots (the autocorrelations and partial autocorrelations), we see that the decay of the autocorrelations are very slow and the seasonality factor seems to be 4. By regressing \( y_t \) on \( y_{t-4} \) we calculate an estimate of variance; \( \sigma_n^2 = 1.0867 \) and using SAS's proc spectra, we calculate the first periodogram ordinate \( I_n(w_1) = 274.56 \). The value of the test statistic \( (T_n(w_1)) \) is 0.99711. According to the rule, we reject the null hypothesis of unit root at \( \alpha = 0.05 \) if \( T_n(w_1) < 0.178 \). And thus, we fail to reject the null hypothesis at 5% level. Table 1 summarizes some other values of the test statistics.

Same procedure is repeated for a stationary seasonal time series \( y_t = 0.8y_{t-4} + e_t \) where \( e_t \) is a sequence of independent normally distributed random variables with mean zero and variance 1. From identification plots (the autocorrelations and partial autocorrelations), we see that the decay of the autocorrelations are very fast and seasonality factor seems to be 4. The value of the first periodogram ordinate is 1.60598 and the value of the test statistic is 0.005244 which is smaller that the 5% critical value 0.178 so that we reject the null hypothesis of a unit root.

Table 1. Values of the Test Statistics

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \rho = 1 )</th>
<th>( \rho = 1 )</th>
<th>%5 Critical Value</th>
<th>( \rho = 0.8 )</th>
<th>( \rho = 0.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq.</td>
<td>0.06283</td>
<td>0.25133</td>
<td>0.178</td>
<td>0.06283</td>
<td>0.25133</td>
</tr>
<tr>
<td>( I_n(w_k) )</td>
<td>274.56</td>
<td>17.924</td>
<td>0.178</td>
<td>1.60598</td>
<td>2.47794</td>
</tr>
<tr>
<td>( T_n(w_k) )</td>
<td>0.99711</td>
<td>1.03638</td>
<td>0.178</td>
<td>0.00524</td>
<td>0.1289</td>
</tr>
</tbody>
</table>

- Fail to Reject \( H_0 \)
- Fail to Reject \( H_0 \)
- Reject \( H_0 \)
- Reject \( H_0 \)

Some other critical values of the test statistics are summarized in Table 2 which can be used for calculation of the powers.
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Table 2. Percentiles of \( Z_1^2 + 3Z_2^2 \) (\( Z_i \) independent \( N(0, 1) \))

\[
\begin{bmatrix}
\alpha & 0.001 & 0.01 & 0.025 & 0.05 & 0.10 & 0.20 & 0.50 & 0.80 & 0.90 & 0.95 & 0.975 & 0.99 \\
z & 0.035 & 0.0348 & 0.088 & 0.178 & 0.368 & 0.79 & 2.54 & 6.32 & 9.48 & 12.85 & 16.37 & 21.17
\end{bmatrix}
\]

Conclusion

In this study, the distribution of the periodogram ordinates of seasonal time series with a unit root has been derived. Using the distributional properties of the periodogram ordinate under the null and alternative hypothesis have been discussed and it is shown that the testing procedure given in Akdi and Dickey (1998) can be applied to the seasonal time series with a unit root. For an illustration, two data sets generated from a seasonal time series are discussed.

References


ÖZET

Bu çalışma periodogram ile yapılan birim kök testlerini mevsimsel zaman serilerine genişletmektedir. Birim köklü mevsimsel zaman serilerinde birim kökler tekrar etmektedir. Burada gösterilmiştir ki, birim köklü AR se'leri için elde edilen normalleştirilmiş periodogramların asimptotik dağılımları birim köklü mevsimsel zaman serileri için de aynıdır.