A CHARACTERIZATION OF UNIFORM DISTRIBUTION

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Abstract

Let \( X_1, X_2, ..., X_n \) be a random sample from a population with probability density function (p.d.f.) \( f(x), (x > 0) \), and let \( Y_{(i)} = \sum_{i=1}^{n} \frac{X_i}{X_{(n)}} \), where \( X_{(n)} = \max_{1 \leq i \leq n} X_i \). A necessary and sufficient condition based on the statistic \( Y_{(i)} \) that an absolutely continuous (with respect to Lebesque measure) p.d.f. \( f(x), x > 0 \), will be rectangular, is given.

Key Words: Characterization, uniform distribution, Mellin transform

1. Introduction

Stapleton (1963) gave a characterization of the uniform distribution on a compact topological group. His results are as follows: Let \( X_1, X_2, ..., X_n \) be \( n \) independent random variables taking values in a compact, separable, connected commutative group \( \Gamma \) such that \( X_j \) takes for no \( j \) all its values in a fixed coset of a proper compact subgroup of \( \Gamma \). Let \( A = (a_{ij}) \) be an \( n \times n \) matrix of integers such that for each \( i \) at least two \( a_{ij} \)'s are different from zero and let \( \det A = \pm 1 \). Suppose that the distribution of \( X_i (i = 1, 2, ..., n) \) has an absolutely continuous component with respect to Haar measure on \( \Gamma \). Let

\[
Z_i = \sum_{j=1}^{n} a_{ij} X_j \quad (i = 1, 2, ..., n);
\]

(1.1)

if \( Z_1, Z_2, ..., Z_n \) are independent then each \( X_j \) is uniformly distributed in \( \Gamma \). Conversely, if \( X_1, X_2, ..., X_n \) are independent uniformly distributed random variables with
values in a connected group $\Gamma$ and if $(a_{ij})$ is a matrix of integers, then $Z_1, Z_2, ..., Z_n$ defined by (1.1) are independently and uniformly distributed if, and only if, $(a_{ij})$ is non-singular.

In the present paper we consider the characterization problem based on the statistics

$$\sum_{i=1}^{n} \frac{X_i}{\max(X_1, X_2, ..., X_n)}.$$

2. Main Result

**Theorem 1.** Let $X$ be a positive random variable having a non-decreasing absolutely continuous probability distribution function $F$. Then $X$ has probability density function

$$f(x) = \begin{cases} 1/a, & 0 < x \leq a \\ 0, & \text{otherwise} \end{cases}$$

if and only if

$$Y_{(1)} = \sum_{i=1}^{n} \frac{X_i}{\max(X_1, X_2, ..., X_n)} \overset{d}{=} 1 + \sum_{i=1}^{n-1} U_i$$

for some two consecutive values of $n = m$ and $m + 1$, where $U_i$ ($i = 1, 2, ..., n - 1$) are independent and uniformly distributed over $(0, 1]$ r.v.'s and $m(>1)$ is some integer. In (2.1) $a$ is an arbitrary positive number.

**Proof.** The necessity of condition (2.2) is trivially established by considering the characteristic function of $Y_{(1)}$ (see Darling (1952)) given by

$$\Phi_n(t) = E\left(e^{itY_{(1)}}\right) = n e^{it} \int_0^\infty \left[ \int_0^1 e^{it\alpha} f(\alpha\beta) d\alpha \right]^{n-1} f(\beta) d\beta. \quad (2.3)$$

Substituting for $f(x)$ as given in (2.1), we obtain

$$\Phi_n(t) = e^{it} \left( e^{it} - \frac{1}{it} \right)^{n-1} \quad (2.4)$$

But the characteristic function of $1 + \sum_{i=1}^{n-1} U_i$ is $e^{it} \left( e^{\frac{it}{it}} - \frac{1}{it} \right)^{n-1}$.

To prove the sufficiency we must prove that the integrofunctional equation

$$n \int_0^\infty \left[ \int_0^1 e^{it\alpha} f(\alpha\beta) d\alpha \right]^{n-1} f(\beta) d\beta = \left( e^{it} - \frac{1}{it} \right)^{n-1}. \quad (2.5)$$

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has the unique solution (2.1) (except for the arbitrariness of the positive constant \( a \)) for some two consecutive values of \( n = m \) and \( m + 1 \) and all real \( t \).

Let us write the left hand side of (2.5) as a multiple integral

\[
\int_0^\infty d\beta f(\beta) \beta^{n-1} \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{n-1} e^{it(\alpha_1 + \cdots + \alpha_{n-1})} f(\alpha_1 \beta) \cdots f(\alpha_{n-1} \beta)
\]

Considered as a Lebesque integral, the interchange of \( \beta \) and \( \alpha \)-integrals is justified since \( f(x) \) is a non-negative function for \( 0 < x < \infty \). Hence we can write this integral as

\[
\int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{n-1} e^{it(\alpha_1 + \cdots + \alpha_{n-1})} \int_0^\infty d\beta \beta^{n-1} f(\beta) f(\alpha_1 \beta) \cdots f(\alpha_{n-1} \beta).
\]

(2.6)

The right hand side of (2.5) is obviously equal to

\[
\int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{n-1} e^{it(\alpha_1 + \cdots + \alpha_{n-1})}.
\]

(2.7)

Hence the functional equation (2.5) reduces to

\[
\int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{n-1} e^{it(\alpha_1 + \cdots + \alpha_{n-1})} G(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 0,
\]

(2.8)

where

\[
G(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \equiv \int_0^\infty d\beta \beta^{n-1} f(\beta) f(\alpha_1 \beta) \cdots f(\alpha_{n-1} \beta) - \frac{1}{n}.
\]

(2.9)

We may assume that \( G(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \) satisfies a Dirichlet condition (see Sneddon (1951)) in each of the \( \alpha \)-variables in the \( n - 1 \) dimensional box \( 0 < \alpha_i \leq 1 \) \( (i = 1, 2, \ldots, n - 1) \). Let us define a function \( H(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \) on \( -\infty < \alpha_i < \infty \) \( (i = 1, 2, \ldots, n - 1) \) such that

\[
H(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \begin{cases} 
0, & -\infty < \alpha_i < 0 \\
G(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}), & 0 < \alpha_i \leq 1, \ (i = 1, 2, \ldots, n - 1) \\
0, & 1 < \alpha_i < \infty
\end{cases}
\]

(2.10)

Then \( H(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \) certainly satisfies a Dirichlet condition in each \( \alpha_i \) on the whole space. We then have an 'enlarged' equation

\[
\int_{-\infty}^\infty d\alpha_1 \cdots \int_{-\infty}^\infty d\alpha_{n-1} e^{it(\alpha_1 + \cdots + \alpha_{n-1})} H(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 0
\]

(2.11)

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By the uniqueness theorem on Fourier transforms, the only solution of (2.11) is the trivial solution

\[ H(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 0 \text{ for } -\infty < \alpha_i < \infty, \ (i = 1, 2, \ldots, n - 1), \]

which implies

\[ G(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = 0 \text{ for } 0 < \alpha_i \leq 1, \ (i = 1, 2, \ldots, n - 1) \] (2.12)

almost everywhere.

Hence we are lead to the functional equation

\[ \int_0^\infty d\beta \beta^{n-1} f(\beta) f(\alpha_1 \beta) \cdots f(\alpha_{n-1} \beta) = \frac{1}{n}, \quad n = m, \ m + 1, \] (2.13)

where all \( \alpha_i \)'s are arbitrary except for the restriction \( 0 < \alpha_i \leq 1, \ (i = 1, 2, \ldots, n - 1) \).

In particular, if we set \( \alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 1 \), then we get

\[ \int_0^\infty d\beta \beta^{n-1} f^n(\beta) = \frac{1}{n} \] (2.14)

for \( n = m \) and \( m + 1 \).

Convergence of the integral in (2.14) requires that

\[ \beta f(\beta) \to 0 \text{ as } \beta \to 0^+ \] (2.15)

and

\[ \beta f(\beta) \to 0 \text{ as } \beta \to \infty. \] (2.16)

Expect for the exponent \( n \) in \( f^n(\beta) \) in equation (2.14) we have a situation similar to the Mellin transform and it is well-known that the inverse Mellin transform of \( 1/n \) is (see Erdelyi (1954)) \( g(\beta) \), given by

\[ g(\beta) = \begin{cases} 1, & 0 < \beta < 1, \\ 0, & \beta > 1. \end{cases} \] (2.17)

In fact, this is also a solution of (2.14) as can be easily verified. A slight generalization of (2.17) satisfying (2.15) and (2.16) is given by

\[ \delta_a(\beta) = \begin{cases} 1/\alpha, & 0 < \beta \leq \alpha, \\ 0, & \beta > \alpha. \end{cases} \] (2.18)

This also satisfies equation (2.14).
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To prove the uniqueness, let us set

\[ f(\beta) = \delta_a(\beta) + h(\beta), \]

where \( h(\beta) \) is independent of \( a \) except for the constraint that \( h(\beta) + \delta_a(\beta) \geq 0 \). Since

\[ F(\infty) = \int_0^\infty f(\beta)d\beta = 1, \quad \int_0^\infty \delta_a(\beta)d\beta = 1, \]

we have

\[ \int_0^\infty d\beta h(\beta) = 0. \tag{2.19} \]

Also, from (2.15) it follows that

\[ \beta h(\beta) \to 0 \text{ as } \beta \to 0^+. \tag{2.20} \]

Now

\[ f^n(\beta) = \sum_{k=0}^n \binom{n}{k} \delta_a^{n-k}(\beta) h^k(\beta). \]

Hence

\[ 1/n = \sum_{k=0}^n \binom{n}{k} \int_0^\infty d\beta \beta^{n-1} \delta_a^{n-k}(\beta) h^k(\beta) \]

\[ = \frac{1}{n} + \int_0^\infty d\beta \beta^{n-1} h^n(\beta) + \sum_{k=1}^{n-1} \binom{n}{k} \int_0^\infty d\beta \beta^{n-1} \delta_a^{n-k}(\beta) h^k(\beta). \tag{2.21} \]

A typical integral on the right in the third term is

\[ \int_0^\infty d\beta \beta^{n-1} \delta_a^{n-k}(\beta) h^k(\beta), \quad 1 \leq k \leq n - 1 \]

\[ = \int_0^a d\beta \beta^{n-1} h^k(\beta)/a^{n-k} \tag{2.22} \]

If one lets \( a \) approach zero from the right, then, in view of (2.20) this integral approaches the limit

\[ a^{n-1} h^k(a)/(n-k)a^{n-k-1} \to 0. \tag{2.23} \]

Since in (2.21) \( a(>0) \) is arbitrary and \( h(\beta) \) is independent of \( a \), (2.23) implies that

\[ \int_0^\infty d\beta \beta^{n-1} h^n(\beta) = 0 \tag{2.24} \]

for \( n = m \) and \( n = m + 1 \), where \( m (>1) \) is some integer.
Rewriting (2.24) for \( n = m \) and \( n = m + 1 \) we have

\[
\int_{0}^{\infty} d\beta \beta^{m-1} h^m(\beta) = 0, \tag{2.25}
\]

\[
\int_{0}^{\infty} d\beta \beta^m h^{m+1}(\beta) = 0. \tag{2.26}
\]

Using the equations (2.25) and (2.26), one can write

\[
\int_{0}^{\infty} d\beta \beta^{m-1} h^m(\beta) [\beta h(\beta) - 1] = 0. \tag{2.27}
\]

If \( h(\beta) \geq 0 \), then from (2.25) and (2.26) it is obvious that \( h(\beta) = 0 \). Suppose \( h(\beta) \leq 0 \) for some \( \beta \) and \( m \) is odd, then from (2.27) evidently \( \beta h(\beta) \geq 1 \), i.e. \( h(\beta) \geq \frac{1}{\beta} \) which is a contradiction (here \( \beta > 0 \)) unless \( h(\beta) = 0 \). Suppose \( h(\beta) \leq 0 \) for some \( \beta \) and \( m \) is even, then from (2.27) it is obvious that \( \beta h(\beta) \leq 1 \), i.e. \( h(\beta) \leq 0 \). Since \( m \) is even it follows from (2.25) that \( h(\beta) = 0 \). Thus

\[
h(\beta) = 0 \text{ for all } \beta \text{ in } (0, \infty). \tag{2.25}
\]

This proves that (2.18) is the unique solution of (2.14). It can be easily verified that (2.18) is also a solution for the more general equation (2.13). But, \( \alpha_i \)'s are quite arbitrary in the domain \( (0, 1] \) and the right hand side of (2.13) has no dependence on the \( \alpha \)'s. Hence (2.18) must also be the unique solution of (2.13).

**A note for applications.** Consider a technical system \( A \) consisting of \( n \) independent components. Let \( X_i \) (\( i = 1, 2, \ldots, n \)) be the survival time of the \( i \)th component. Then \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) are the successive failure times. Let \( Y_1 = X_1 - X_{(1)}, Y_2 = X_2 - X_{(1)}, \ldots, Y_n = X_n - X_{(1)} \) and denote by \( Y'_1, Y'_2, \ldots, Y'_{n-1} \) \( n - 1 \) variables among \( Y_1, Y_2, \ldots, Y_n \) which do not vanish. It is not difficult to prove that if \( X_1, X_2, \ldots, X_n \) are exponentially distributed i.i.d. random variables with d.f. \( F(x) = 1 - \exp(-\lambda x), \)
\( x \geq 0, \lambda > 0 \), then \( Y'_1, Y'_2, \ldots, Y'_{n-1} \) also are i.i.d. random variables having the same exponential distribution. Let \( N(t) \) be the number of failures at time \( t \) and \( N(0) = 0 \). One write \( N(t) = k \) iff \( X_{(k)} \leq t < X_{(k+1)} \). It is clear that system \( A \) will stay at state \( N(t) = 0 \) for a random time \( Y'_0 = X_{(1)} \) and after this time system will change its state to \( N(t) = 1 \). Staying in state \( N(t) = 1 \) for a random time \( Y'_{(1)} \) the system will change its state to \( N(t) = 2 \) etc. Denoting by \( Y''_1, Y''_2, \ldots, Y''_{n-2} \) \( n - 2 \) variables among \( Y'_1 - Y'_{(1)}, Y'_2 - Y'_{(1)}, \ldots, Y'_{n-1} - Y'_{(1)} \) which do not vanish we can conclude that \( Y''_1, Y''_2, \ldots, Y''_{n-2} \) are i.i.d. with the same exponential distribution etc. It is not difficult to observe that Theorem 1 can be rewritten as follows:
Corollary. Let $X_i (i=1,2,...,n)$ be i.i.d. nonnegative continuous random variables with d.f. $F(x)$. Let $Y_i (i=1,2,...,n)$ and $Y'_1,Y'_2,...,Y'_{n-1}$ are defined as above. Then $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$ for some $\lambda > 0$ if and only if
\[
1 + \sum_{i=1}^{n-1} Y'_i \overset{d}{=} 1 + \sum_{i=1}^{n-1} Z_i,
\]
for two consecutive values of $n = m$ and $m + 1$, where $m (> 1)$ is some integer and $Z_i, i=1,2,...,n-1$ are i.i.d. random variables with d.f. $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0, \lambda > 0$.

References


ÖZET

$X_1, X_2, ..., X_n$ bağımsız ve aynı mutlak sürekli $F$ dağılımına sahip olan rasgele değişkenler olmak üzere $X_{(n)} = \max_{1 \leq i \leq n} X_i$ ve $Y_{(1)} = \sum_{i=1}^{n} \frac{X_i}{X_{(n)}}$ olsun. $F$ in $[0,1]$ de düzgün dağılım fonksiyonu olması için $Y_{(1)}$ istatistiğine dayalı olarak gereklidir ve yeter koşul verilmiştir.