CONCOMITANT OF ORDER STATISTICS IN FGM TYPE BIVARIATE UNIFORM DISTRIBUTIONS

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Abstract

We consider the bivariate FGM distributions with uniform marginals. The distribution of the concomitant of the \( r \) th order statistic of one of the component is obtained. Recurrence relations between moments of concomitants are given.

Key Words: Farlie-Gumbel-Morgenstern distributions, order statistics, norm, concomitants, recurrence relations, product moments.

1. Introduction

The class of bivariate distributions originally proposed by Morgenstern (1956) having a natural form

\[
F_{X,Y}(x,y) = F_X (x) F_Y (y) \left[ 1 + \alpha [1 - F_X (x)] [1 - F_Y (y)] \right]
\]  \hspace{1cm} (1.1)

is a flexible family useful in applications provided the correlation between the variables is not too large. It can be utilized for arbitrary continuous marginals. This structure was studied by Farlie (1960) in the form (FGM) of

\[
F_{X,Y}(x,y) = F_X (x) F_Y (y) \left[ 1 + \alpha A (F_X (x)) B (F_Y (y)) \right]
\]  \hspace{1cm} (1.2)

where \( A(x) \) and \( B(y) \) satisfy certain regularity conditions ensuring that (1.2) is a distribution function with absolutely continuous marginals \( F_X (x) \) and \( F_Y (y) \).
Further generalizations of (1.1) to distributions with more than two variables and a stronger correlation structure can be found in Johnson and Kotz (1975, 1977), Kotz and Johnson (1977) and Huang and Kotz (1984). Recent results dealing with this family of distributions are due to Huang and Kotz (1998) who introduce an additional parameter to increase the dependence between the underlying variables. Bairamov and Kotz (1999) present several theorems characterizing symmetry and dependence properties of FGM and Huang-Kotz FGM distributions and provide a modification of Huang-Kotz FGM distributions with large correlation between the components.

Let \((X_i, Y_i), i = 1, 2, ..., n\) be a random sample from an absolutely continuous bivariate population \((X, Y)\) with distribution function (d.f.) \(F_{X,Y}(x,y)\). Let \(X_{r:n}\) denote the \(r\)th order statistics of the \(X\) sample values. Denote by \(Y_{r:n}\) the \(Y\) values associated with \(X_{r:n}\). We call \(Y_{r:n}\) the concomitant of the \(r\)th order statistic. Concomitants are used, for example, in selection procedures. Recently Balasubramanian and Beg (1998) have studied concomitants in Gumbel’s bivariate exponential distribution. For more details we refer to the review articles of Bhattacharya (1984) and David (1993). Denote probability density function (p.d.f.) of \(Y_{r:n}\) by \(g_{r:n}(y)\). It is known that

\[
g_{r:n}(y) = \int_{-\infty}^{+\infty} f(y \mid x) f_{r:n}(x) \, dx, \tag{1.3}
\]

where \(f(y \mid x)\) is the condition density function of \(Y\), given \(X\) and \(f_{r:n}(x)\) is the p.d.f. of \(X_{r:n}\) (see David (1981), p.110.).

In this paper we shall consider the classical Morgenstern distribution (1.1) with uniform marginals and investigate the distributional and moment properties of concomitants of order statistics.

2. Concomitants in Morgenstern type bivariate distributions

Consider (1.1) for \(F_X(x) = x\), \(F_Y(y) = y\); \(0 < x, y < 1\).

\[
F(x, y) = xy \{1 + \alpha (1 - x)(1 - y)\}, \quad -1 \leq \alpha \leq 1. \tag{2.1}
\]

and

\[
f(x, y) = 1 + \alpha (1 - 2x)(1 - 2y), \quad 0 \leq x, y \leq 1. \tag{2.2}
\]

The d.f. of \(Y_{r:n}\) is given by

\[
G_{r:n}(y) = \int_{-\infty}^{+\infty} F(y \mid x) f_{r:n}(x) \, dx, \tag{2.3}
\]

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where \( F(y \mid x) \) is a conditional d.f. of \( Y \) given \( X \), and

\[
f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x)
\]

is the p.d.f. of \( r \) th order statistic. Since the marginals of (2.1) are Uniform \([0,1]\) we arrive at

\[
F(y \mid x) = y \left\{ 1 + \alpha (1 - 2x) (1 - y) \right\} \quad \text{and} \quad f(y \mid x) = 1 + \alpha (1 - 2x) (1 - 2y) \quad (2.4)
\]

Using (2.3) and (2.4) we derive the d.f. of \( Y_{r:n} \):

\[
G_{r:n}(y) = \int_0^1 y \left\{ 1 + \alpha (1 - 2x) (1 - y) \right\} \frac{n!}{(r-1)!(n-r)!} x^{r-1} (1 - x)^{n-r} \, dx = \frac{n!}{(r-1)!(n-r)!} y \left\{ 1 + \alpha \left[ 1 - 2 \frac{r}{n+1} \right] (1 - y) \right\} \quad (2.5)
\]

and the corresponding p.d.f. is

\[
g_{r:n}(y) = 1 + \alpha \left[ 1 - 2 \frac{r}{n+1} \right] (1 - 2y) \quad (2.6)
\]

Consider the moments of \( Y_{r:n} \). From (2.6), the \( k \) th moment of \( Y_{r:n} \) is given by

\[
\mu_{r:n}^{(k)} = E\left\{ Y_{r:n}^k \right\} = \int_0^1 y^k \left\{ 1 + \alpha \left[ 1 - 2 \frac{r}{n+1} \right] (1 - 2y) \right\} \, dy = \frac{1}{k+1} \left\{ 1 - \alpha \left[ 1 - 2 \frac{r}{n+1} \right] \left( \frac{k}{k+2} \right) \right\}, \quad k = 0, 1, 2, \ldots \quad (2.7)
\]

Consequently the expected value and the variance of \( Y_{r:n} \) can be obtained from (2.7) as follows:

\[
E\left\{ Y_{r:n} \right\} = \frac{1}{2} \left\{ 1 - \frac{\alpha}{3} \left[ 1 - 2 \frac{r}{n+1} \right] \right\} \quad (2.7a)
\]

and

\[
Var\left\{ Y_{r:n} \right\} = \frac{1}{12} \left\{ 1 - \frac{\alpha^2}{3} \left[ 1 - 4 \frac{r}{n+1} \left( 1 - \frac{r}{n+1} \right) \right] \right\}. \quad (2.7b)
\]

The moment generating function (m.g.f.) of \( Y_{r:n} \) is given by

\[
M_{r:n}(t) = E\left\{ e^{tY_{r:n}} \right\} = \int_0^1 e^{ty} \left\{ 1 + \alpha \left[ 1 - 2 \frac{r}{n+1} \right] (1 - 2y) \right\} \, dy = \frac{e^t - 1}{t} \left\{ 1 + \alpha \left[ 1 - 2 \frac{r}{n+1} \right] \left[ 1 + 2 \left( \frac{1}{t} - \frac{e^t}{e^t - 1} \right) \right] \right\} \quad (2.8)
\]
3. Recurrence relation between moments of concomitants

From (2.6) one observes \( g_{[r:n-1]}(y) = 1 + \alpha \left[ 1 - 2 \frac{r}{n} \right] (1 - 2y) \) and

\[
g_{[r:n]}(y) - g_{[r:n-1]}(y) = \alpha \frac{2r}{n(n+1)} (1 - 2y)
\]

(3.1)

Also

\[
g_{[r-1:n]}(y) = 1 + \alpha \left[ 1 - 2 \frac{r-1}{n+1} \right] (1 - 2y)
\]

and

\[
g_{[r:n]}(y) - g_{[r-1:n]}(y) = -\alpha \frac{2}{n+1} (1 - 2y).
\]

(3.2)

Relation (3.1) and (3.2) can be extended:

\[
g_{[r:n]}(y) - g_{[r:n-i]}(y) = \alpha \frac{2ri}{(n+1)(n-i+1)} (1 - 2y), \quad 1 \leq i \leq n-r
\]

(3.3)

and

\[
g_{[r:n]}(y) - g_{[r-j:n]}(y) = -\alpha \frac{2j}{(n+1)} (1 - 2y), \quad 1 \leq j \leq r-1
\]

(3.4)

Moreover the following equalities are valid

\[
g_{[r-j:n-i]}(y) = 1 + \alpha \left[ 1 - 2 \frac{r-j}{n-i+1} \right] (1 - 2y), \quad 1 \leq i \leq n-r ; 1 \leq j \leq r-1
\]

and

\[
g_{[r:n]}(y) - g_{[r-j:n-i]}(y) = \alpha \frac{2[ri - j(n+1)]}{(n+1)(n-i+1)} (1 - 2y), \quad 1 \leq i \leq n-r ; 1 \leq j \leq r-1
\]

(3.5)

Let \( 1 \leq i_1 < i_2 \leq n-r \) and \( 1 \leq j_1 < j_2 \leq r-1 \). Then

\[
g_{[r-j_1:n-i_1]}(y) - g_{[r-j_2:n-i_2]}(y) = 2\alpha \left[ \frac{r-j_2}{n-i_2+1} - \frac{r-j_1}{n-i_1+1} \right] (1 - 2y)
\]

(3.6)

Using (3.6) one obtains the following general recurrence relation between the moments of concomitants:

\[
\mu_{[r-j_1:n-i_1]}^{(k)} - \mu_{[r-j_2:n-i_2]}^{(k)} = 2\alpha \left[ \frac{r-j_1}{n-i_1+1} - \frac{r-j_2}{n-i_2+1} \right] \frac{k}{(k+1)(k+2)}
\]

(3.7)

In particular the following relations are valid

\[
\mu_{[r:n]}^{(k)} - \mu_{[r:n-i]}^{(k)} = \alpha \frac{2ri}{(n+1)(n-i+1)} \frac{-k}{(k+1)(k+2)}
\]

(3.8)
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(3.9)

\[ \mu_{[k]}^{(k)} - \mu_{[j]}^{(k)} = \alpha \frac{2j}{(n + 1)(k + 1)(k + 2)} \]

(3.10)

\[ \mu_{[j]}^{(k)} - \mu_{[j-1]}^{(k)} = \alpha \frac{2[ri - j(n + 1)]}{(n + 1)(n - i + 1)(k + 1)(k + 2)} \]

From (2.8) clearly follows the following recurrence relation for m.g.f. of concomitants:

(3.11)

\[ M_{[j_1]}^{(r_n-1)}(t) - M_{[j_2]}^{(r_n-1)}(t) = 2\alpha \left[ \frac{r - j_2}{n - i_2 + 1} - \frac{r - j_1}{n - i_1 + 1} \right] \times \]

\[ \times \frac{e^t - 1}{t} \left[ 1 + 2 \left( \frac{1}{t} - \frac{e^t}{e^t - 1} \right) \right], \quad 1 \leq i_1 < i_2 \leq n - r; \quad 1 \leq j_1 < j_2 \leq r - 1. \]

In particular, one can obtain the following relations between m.g.f. \( Y_{[r]} \) and \( Y_{[j]} \):

(3.12)

\[ M_{[r]}^{(n-i)} = \alpha \frac{2ri}{(n + 1)(n - i + 1)} \frac{e^t - 1}{t} \left[ 1 + 2 \left( \frac{1}{t} - \frac{e^t}{e^t - 1} \right) \right] \]

(3.13)

\[ M_{[r]}^{(n-1)} - M_{[j]}^{(n-r)} = -\alpha \frac{2j}{(n + 1)} \frac{e^t - 1}{t} \left[ 1 + 2 \left( \frac{1}{t} - \frac{e^t}{e^t - 1} \right) \right] \]

(3.14)

\[ M_{[r]}^{(n-1)} - M_{[j]}^{(n-1)} = \alpha \frac{2[ri - j(n + 1)]}{(n + 1)(n - i + 1)} \frac{e^t - 1}{t} \left[ 1 + 2 \left( \frac{1}{t} - \frac{e^t}{e^t - 1} \right) \right] \]

4. Joint distribution of concomitants

Let \( Y_{[r]} \), \( Y_{[s]} \), ..., \( Y_{[k]} \) be the concomitants of \( X_{[r]} \), \( X_{[s]} \), ..., \( X_{[k]} \), respectively, where \( 1 \leq r_1 < r_2 < \ldots < r_k \leq n \). The joint probability density function of \( (Y_{[r]}, Y_{[s]}, \ldots, Y_{[k]}) \) is

(4.1)

\[ g_{y_{[r],y_{[s]},\ldots,y_{[k]}}}(y_1, y_2, \ldots, y_k) = \]

\[ = \int_{-\infty}^{x_k} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_1} f(y_1 | x_1) f(y_2 | x_2) \cdots f(y_k | x_k) f_{[r],y_{[s]},\ldots,y_{[k]}}(x_1, x_2, \ldots, x_k) dx_1 \cdots dx_k \]

where \( f_{[r],y_{[s]},\ldots,y_{[k]}}(x_1, x_2, \ldots, x_k) \) is the joint p.d.f. of \( (X_{[r]}, X_{[s]}, \ldots, X_{[k]}) \).

The joint p.d.f. of two concomitants \( Y_{[r]}, Y_{[s]} \) (\( 1 \leq r < s \leq n \)) for (2.1) can be calculated by using (4.1) as follows

\[ g_{Y_{[r]},Y_{[s]}}(y_1, y_2) = 1 + \alpha \left( 1 - 2 \frac{r}{n + 1} \right) (1 - 2y_1) + \alpha \left( 1 - 2 \frac{s}{n + 1} \right) (1 - 2y_2) \]

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\[ +\alpha^2 \left( 1 - 2 \frac{r + s}{n + 1} + 4 \frac{r (s + 1)}{(n + 1) (n + 2)} \right) (1 - 2y_1) (1 - 2y_2) \]  

(4.2)

For \( k = 3 \), the joint p.d.f. of \( (Y_{r_1;n}, Y_{r_2;n}, Y_{r_3;n}) \) \( (1 \leq r_1 < r_2 < r_3 \leq n) \) is

\[
g_{[r_1,r_2,r_3]} (y_1, y_2, y_3) = 1 + \alpha \left( 1 - 2 \frac{r_1}{n + 1} \right) (1 - 2y_1) +
\]

\[
+ \alpha \left( 1 - 2 \frac{r_2}{n + 1} \right) (1 - 2y_2) + \alpha \left( 1 - 2 \frac{r_3}{n + 1} \right) (1 - 2y_3) +
\]

\[
+ \alpha^2 \left( 1 - 2 \frac{r_1 + r_2}{n + 1} + 4 \frac{r_1 (r_2 + 1)}{(n + 1) (n + 2)} \right) (1 - 2y_1) (1 - 2y_2) +
\]

\[
+ \alpha^2 \left( 1 - 2 \frac{r_1 + r_3}{n + 1} + 4 \frac{r_1 (r_3 + 1)}{(n + 1) (n + 2)} \right) (1 - 2y_1) (1 - 2y_3) +
\]

\[
+ \alpha^2 \left( 1 - 2 \frac{r_2 + r_3}{n + 1} + 4 \frac{r_2 (r_3 + 1)}{(n + 1) (n + 2)} \right) (1 - 2y_2) (1 - 2y_3) +
\]

\[
+ \alpha^3 \left( 1 - 2 \frac{r_1 + r_2 + r_3}{n + 1} + 4 \frac{r_1 (r_2 + 1) + r_1 (r_3 + 1) + r_2 (r_3 + 1)}{(n + 1) (n + 2)} \right) (1 - 2y_1) (1 - 2y_2) (1 - 2y_3).
\]

(4.3)

5. Concomitant of the norm-ordered statistics

Bairamov and Gebizlioglu (1998) introduced norm-ordered statistics for multivariate data. Let \( R^m, m \geq 1, \) be the real Euclidean space. Suppose \( X_1, X_2, \ldots, X_n \in R^m \) are independent identically distributed (i.i.d.) random variables \( (m > 1 \) random vectors \( ) \) r.v.’s with distribution function \( d.f. \) \( F \). Denote by \( \| \| \) the norm defined in \( R^m \). It is clear that \( \| X_1 \|, \| X_2 \|, \ldots, \| X_n \| \) are i.i.d. r.v. with d.f. \( P \{ \| X_i \| \leq x \} \equiv F^*(x), x \in R \). If \( F \) is assumed to be continuous, the probability of any two or more of these r.v. assuming equal magnitudes is zero. Therefore, there exists a unique ordered arrangement within the r.v. \( \| X_i \|, i = 1, 2, \ldots, n \). We say that \( X_1 \) precedes \( X_2 \) (or that \( X_1 \) is less than \( X_2 \) in a norm sense) if \( \| X_1 \| \leq \| X_2 \| \) and denote \( X_1 \prec X_2 \). Suppose \( X^{(1)} \) denotes the smallest of the set \( X_1, X_2, \ldots, X_n \); \( X^{(2)} \) denotes the second smallest, etc.; and \( X^{(n)} \) denotes the largest in a norm sense. The distribution of norm-ordered statistics is expressed in terms of the so called structural function \( h(x, y) = P \{ \| X_1 \| \leq \| x \| \}, \) where \( x = (x_1, x_2, \ldots, x_m) \in R^m \) which can be estimated empirically. Specifically the p.d.f. of \( r \) th norm-ordered statistic \( X^{(r)} \) is

\[
f_r(x_1, x_2, \ldots, x_k) =
\]

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\[ n \binom{n-1}{r-1} [1 - h(x_1, x_2, \ldots, x_k)]^{r-1} [1 - h(x_1, x_2, \ldots, x_k)]^{n-r} f(x_1, x_2, \ldots, x_k) \]

\[ if \quad \|\bar{x}_1\| < \|\bar{x}_2\| < \ldots < \|\bar{x}_k\| \]

and

\[ f_r(x_1, x_2, \ldots, x_k) = 0, \quad otherwise \]

The joint p.d.f. of \((X^{(1)}, X^{(2)}, \ldots, X^{(n)})\) is

\[ f_{1,2,\ldots,n}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \]

\[ = \left\{ \begin{array}{ll}
  n! f(x_1, y_1) f(x_2, y_2) \ldots f(x_n, y_n), & if \quad \|\bar{x}_1\| < \|\bar{x}_2\| < \ldots < \|\bar{x}_k\| \\
  0, & otherwise
\end{array} \right. \]

Here we define concomitants for norm-ordered statistics as follows.

Let \((X, Y, Z)\) be the absolutely continuous r.v. with the d.f. \(F(x, y, z)\) and p.d.f. \(f(x, y, z)\). Let \((X_i, Y_i, Z_i), i = 1, 2, \ldots, n\) be the independent copies of \((X, Y, Z)\). Suppose that the first two coordinates of \((X, Y)\) are ordered in a norm sense, i.e., let \(X^{(1)} < X^{(2)} < \ldots < X^{(n)}\) be the norm-ordered statistics of \((X_i, Y_i)\) \(i = 1, 2, \ldots, n\). Denote by \(Z_{[r:n]}\) the Z values associated with \(X^{(r)}\). We call \(Z_{[r:n]}\) the concomitant of \(r\) th norm-ordered statistics \(X^{(r)}\).

The d.f. of \(Z_{[r:n]}\) can be found as follows:

\[ P\{Z_{[r:n]} \leq z\} = P\left\{ \bigcup_{k=1}^{n} Z_k \leq z, X^{(r)} \right\} \]

By using total probability formula for continuous random variables

\[ P(A) = \int_{-\infty}^{+\infty} P(A \mid X = x) dF_X(x) \]

one can write

\[ P\{Z_{[r:n]} \leq z\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(z \mid x, y) f_{r:n}(x, y) \cdot t_y \equiv G_{[r:n]}(z) \quad (5.1) \]

and

\[ g_{[r:n]}(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z \mid x, y) f_{r:n}(x, y) dx dy, \]

where

\[ f_{r:n}(x, y) = \frac{n!}{(r-1)! (n-r)!} [h(x, y)]^{r-1} [1 - h(x, y)]^{n-r} f(x, y) dx dy, \quad (5.2) \]

\[ h(x, y) = P\{\|X, Y\| \leq \|(x, y)\|\} \]

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Now consider the following three-variate FGM distributions with unit exponential marginals:

\[ F(x, y, z) = \left(1 - e^{-x}\right) \left(1 - e^{-y}\right) \left(1 - e^{-z}\right) \left\{1 + \alpha e^{-x-y-z}\right\}, \quad x, y, z > 0; \quad -1 \leq \alpha \leq 1. \quad (5.3) \]

This is a trivariate extension of Gumbel’s bivariate exponential distribution.

The p.d.f. is

\[ f(x, y, z) = e^{-x-y-z} \left\{1 + \alpha \left[2e^{-x} - 1\right] \left[2e^{-y} - 1\right] \left[2e^{-z} - 1\right]\right\}, \quad x, y, z > 0 \quad (5.4) \]

Evidently

\[ f(z \mid x, y) = e^{-z} \left\{1 + \alpha \left[2e^{-x} - 1\right] \left[2e^{-y} - 1\right] \left[2e^{-z} - 1\right]\right\}, \quad x, y, z > 0; \quad (5.5) \]

\[ F(z \mid x, y) = \int_{0}^{z} f(z \mid x, y) \, dz \]

\[ = \left(1 - e^{-z}\right) \left\{1 + \alpha e^{-z} \left[2e^{-x} - 1\right] \left[2e^{-y} - 1\right]\right\} \quad (5.6) \]

By using (5.1) we can obtain the d.f. of norm-ordered concomitants for the distribution of the form (5.3). For example, consider the distribution function of \(Z_{[1,n]}\). Let \(\|(x, y)\| = |x| + |y|\). Then

\[ h(x, y) = P\{|X| + |Y| \leq |x| + |y|\} = P\{X + Y \leq x + y\} = G_{2,1}(x + y). \]

Hence

\[ h(x, y) = 1 - (1 + x + y) e^{-x-y} \quad \text{and} \quad (5.7) \]

\[ 1 - h(x, y) = (1 + x + y) e^{-x-y}. \quad (5.8) \]

Using (5.2)-(5.8) in (5.1) one can write

\[ P\left(Z_{[1,n]} \leq z\right) = \int_{0}^{+\infty} \int_{0}^{+\infty} F(z \mid x, y) n [1 - h(x, y)]^{n-1} f(x, y) \, dx \, dy \]

\[ = \int_{0}^{+\infty} \int_{0}^{+\infty} (1 - e^{-z}) \left\{1 + \alpha e^{-z} \left[2e^{-x} - 1\right] \left[2e^{-y} - 1\right]\right\} \]

\[ n \left[(1 + x + y) e^{-x-y}\right]^{n-1} e^{-x-y} \, dx \, dy \]

\[ = n \left(1 - e^{-z}\right) \int_{0}^{+\infty} \int_{0}^{+\infty} \left\{1 + \alpha e^{-z} \left[2e^{-x} - 1\right] \left[2e^{-y} - 1\right]\right\} (1 + x + y)^{n-1} e^{-nx-ny} \, dx \, dy \]

\[ = n \left(1 - e^{-z}\right) \left\{I_0 + \alpha e^{-z} [4I_1 - 2I_2 - 2I_3 + I_4]\right\}, \]

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where

\[ I_0 = I_4 = \int_0^{+\infty} \int_0^{+\infty} (1 + x + y)^{n-1} e^{-nx-ny} dx dy = \frac{1}{n} , \]

\[ I_1 = \int_0^{+\infty} \int_0^{+\infty} (1 + x + y)^{n-1} e^{-(n+1)x-(n+1)y} dx dy = \frac{1}{n+1} - \frac{(n-1)!}{(n+1)^{n+1}} \sum_{l=0}^{n-1} \frac{(n+1)^l}{l!} , \]

\[ I_2 = \int_0^{+\infty} \int_0^{+\infty} (1 + x + y)^{n-1} e^{-(n+1)x-ny} dx dy = \frac{(n-1)!}{n^n} \sum_{l=0}^{n-1} \frac{n^l}{l!} - \frac{(n-1)!}{(n+1)^n} \sum_{l=0}^{n-1} \frac{(n+1)^l}{l!} , \]

\[ I_3 = \int_0^{+\infty} \int_0^{+\infty} (1 + x + y)^{n-1} e^{-nx-(n+1)y} dx dy = \frac{(n-1)!}{n^n} \sum_{l=0}^{n-1} \frac{n^l}{l!} - \frac{(n-1)!}{(n+1)^n} \sum_{l=0}^{n-1} \frac{(n+1)^l}{l!} , \]

\[ I_2 = I_3 . \]

We thus have

\[ P \{ Z_{[1:n]} \leq z \} = (1 - e^{-z}) \left\{ 1 + \alpha e^{-z} \left[ 4n \left( \frac{1}{n+1} - \frac{(n-1)!}{(n+1)^{n+1}} \sum_{l=0}^{n-1} \frac{(n+1)^l}{l!} \right) - 4n \left( \frac{(n-1)!}{n^n} \sum_{l=0}^{n-1} \frac{n^l}{l!} - \frac{(n-1)!}{(n+1)^n} \sum_{l=0}^{n-1} \frac{(n+1)^l}{l!} \right) + 1 \right] \right\} \]

Denote

\[ h(n) = 1 + 4 - \frac{n}{n+1} + 4nn! \sum_{l=0}^{n-1} \frac{1}{l!} \left[ (n+1)^{l-n-1} - n^{l-n-1} \right] . \]

Then one has

\[ P \{ Z_{[1:n]} \leq z \} = (1 - e^{-z}) \left\{ 1 + \alpha e^{-z} h(n) \right\} \]

and p.d.f. of \( Z_{[1:n]} \)

\[ g_{[1:n]} (z) = \left( 1 - \alpha h(n) e^{-z} \right) + 2\alpha h(n) e^{-2z} , z > 0 . \]  

(5.10)

References


ÖZET

Marjinalleri [0,1] aralığında düzgün dağılım fonksiyonu olan iki değişkenli FGM dağılımları incelenmiştir. İlk bileşenin r-inci sıra istatistiğinin eşini dağılımı elde edilmiştir. Eğlerin momentleri arasındaki indirgeme bağıntıları verilmiştir.