ON THE NUMBER OF PRODUCTIVE ANCESTORS IN LARGE POPULATIONS

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Abstract

We consider a population of $n$ individuals. Each of these individuals generates a discrete time branching stochastic process. We study the number of ancestors $S(n, t)$ whose offspring at time $t$ exceeds level $\theta(t)$, where $\theta(t)$ is some positive valued function. It is proved that $S(n, t)$ may be approximated as $t \to \infty$ by some stochastic processes with independent increments if $n \to \infty$ depending on the time of observation.

Key Words: Population, ancestor, branching process, Poisson process, Brownian motion, Binomial process, exceedance

1. Introduction

We consider a population containing $n$ individuals of the same type at time zero. Each of these individuals (ancestors) initiates a discrete time branching population process. Let $\theta(t), t \in \mathbb{N}_0 = \{0, 1, \ldots\}$ be a positive valued function and $S(n, t)$ be the number of ancestors having more than $\theta(t)$ descendents at time $t$.

Branching processes started by the initial ancestors may be considered as population processes describing population growth in different regions of an area $R$. Then it is easy to see that $S(n, t)$ is the number of regions of $R$ whose population at time $t$ exceeds level $\theta(t)$. Process $S(n, t)$ can be associated with a problem on the number of vertexes of rooted random trees as well. In fact each realization of the scheme under the consideration can be interpreted as a forest containing $n$ rooted trees. Consequently a realization of $S(n, t)$ is the number of trees in the forest having more than $\theta(t)$ vertexes of the level $t$.

We note here the rise of interest in recent years to problems concerning extrema in branching stochastic processes. For example the recent publications in this direction have been devoted to the asymptotic behaviour of the expectation of the maxima of branching processes (Borovkov, Vatutin (1996), Pakes (1998)), to the limit distribution
for the maximum family size (Arnold, Villasenor (1996), Rahimov, Yanev (1999)) and to other problems. Limit distributions for the index of the first process in a sequence of branching processes exceeding some fixed or increasing levels were obtained in Rahimov, Hasan (1998). Thus the study of $S(n, t)$ can be considered as a contribution to this program of investigation of the extrema in population processes.

It follows from well-known properties of branching processes (see Athreya, Ney (1972), for example) that if $n$ is fixed and the process is critical or subcritical, then $S(n, t)$ in the long run equals to zero with probability 1, for any level function $\theta(t)$.

What happens if the size of the initial population is large? In other words what is the asymptotic behaviour of $S(n, t)$ if the number of initial ancestors increases depending on the time of observation? To answer these questions we consider family of stochastic process $y(x, t) = S([m(t)x], t)$, where $x \in [0, \infty)$ and $m(t) \to \infty$ as $t \to \infty$. We approximate $y(x, t)$ by some known processes with independent increments. Behaviour of the parameter $m(t)$ and the form of limit processes naturally depend on criticality of the initial branching process. It turns out that, if the process is supercritical, then $y(x, t)$ may be approximated by either a “binomial process” (process with independent and binomially distributed increments) or by the Brownian motion depending on the behaviour of $m(t)$. If the process is subcritical or critical, then the approximating process is either a Poisson process or the Brownian motion.

Now we give a rigorous definition of the process $S(n, t)$. Let $A^t_i$ be the random population at time $t$ generated by $i$-th initial ancestor, $i = 1, 2, \ldots, n$. For any positive valued function $\theta(t)$ functional $S(n, t) = S(n, t)[\theta]$ can be defined as following

$$S(n, t) = \#\{i : \text{card } A^t_i > \theta(t)\}.$$ 

Let $X_i(t) = \text{card } A^t_i$ be $i$-th branching process and $X(t)$ be a branching process such that $X(t) \overset{d}{=} X_i(t)$ for all $i \geq 1$. We denote $\{P_k, k \geq 0\}$ the offspring distribution of $X(t)$ and put

$$f(S) = \sum_{k=0}^{\infty} P_k S^k, \quad R(x, t) = P\{X(t) > x\}, Q(t) = R(0, t),$$

$$A = \sum_{k=1}^{\infty} k P_k, \quad \sigma^2 = \sum_{k=1}^{\infty} k(k - 1) P_k.$$ 

2. Critical processes

First we consider the critical case, i.e., the case of $A = 1$, $0 < \sigma^2 < \infty$. We assume that there exits the following

$$\lim_{t \to \infty} \frac{\theta(t)}{t} = \theta \in [0, \infty]$$ (1)

and consider $y(x, t) = S([tx], t)$, i.e., $m(t) = t$.

**Theorem 1.** If $A = 1$, $0 < \sigma^2 < \infty$ and (1) is satisfied, then $y(x, t) \overset{D}{\to} y(x)$ as $t \to \infty$, where $D$ means convergence in the weak sense and $y(x)$ is the Poisson process with $E_y(x) = 2x \exp(-20/\sigma^2)/\sigma^2$ for $\theta \in [0, \infty)$ and it is a "zero process" (i.e., $y(x) \equiv 0$ with probability 1 for all $x \in [0, \infty)$) for $\theta = \infty$.

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Theorem 1 gives an approximation of \( S(n, t) \) for the case when \( n = o(t) \) or \( n \approx t \) as \( t \to \infty \). Now we consider the case when \( n/t \to \infty \), \( t \to \infty \). More precisely we put \( m(t) = a(t)t \), where \( a(t) \to \infty \). We define the stochastic process \( W^{(1)}(x) \) as follows

\[
W^{(1)}(x) = \frac{S([ta(t)x], t) - [ta(t)x]R(\theta(t), t)}{\sqrt{a(t)}},
\]

where \( R(\theta(t), t) = P\{X(t) > \theta(t)\}, \ x \in [0, \infty) \).

**Theorem 2.** If \( A = 1, 0 < \sigma^2 < \infty \) and (1) is satisfied, then \( W^{(1)}(x) \Rightarrow W^{(1)}(x) \) as \( t \to \infty \), where \( \Rightarrow \) means, as before, convergence in the weak sense and \( W^{(1)}(x) \) is the Brownian motion with zero shift and with the diffusion parameter \( 2\sigma^{-2} \exp\{-2\theta/\sigma^2\} \) for \( \theta \in [0, \infty) \) and it is a zero process for \( \theta = \infty \).

**Proof of Theorem 1.** Since the lives of individuals are independent and identically distributed we obtain that

\[
ES^{S(n,t)} = (1 - (1 - S)R(\theta(t), t))^n.
\]

We use the following well known results for critical branching processes (see Harris (1963), pp. 19-22). If \( A = 1, 0 < \sigma^2 < \infty \), then for any fixed \( x > 0 \)

\[
P\{Q(t)X(t) > x|X(t) > 0\} \sim e^{-x}, \ Q(t) \sim 2/\sigma^2 t, \ \text{as} \ t \to \infty.
\]

It follows from (3) that under the condition (1)

\[
R(\theta(t), t) \sim \frac{2}{\sigma^2 t} \exp\left\{-\frac{2\theta}{\sigma^2}t\right\} \to \infty.
\]

Therefore

\[
\lim_{t \to \infty} \ln ES^{y(x,t)} = -\lim_{t \to \infty} [xt]R(\theta(t), t)(1 - s)
\]

\[
= -\frac{2x}{\sigma^2}e^{-2\theta/\sigma^2}(1 - s)
\]

for \( \theta \in [0, \infty) \). Consequently the generating function of \( y(x, t) \) tends as \( t \to \infty \) to

\[
\exp\left\{\frac{2x}{\sigma^2}e^{-2\theta/\sigma^2}(s - 1)\right\}
\]

which is the generating function of the one dimensional distribution for Poisson process \( y(x) \).

Now we consider

\[
P\{y(x_i, t) = k_i, \ i = 0, 1, \ldots, r\},
\]

where \( 0 = x_0 < x_1 < \cdots < x_r < \infty, \ r = 1, 2, \ldots \) First we prove that

\[
\lim_{t \to \infty} ES^{y(x_2,t)-y(x_1,t)} = \exp\left\{\frac{2(x_2 - x_1)}{\sigma^2}e^{-2\theta/\sigma^2}(s - 1)\right\}.
\]
In fact, since

\[ y(x_2, t) - y(x_1, t) = \sum_{i=[x_1 t]+1}^{[x_2 t]} \varepsilon_i, \quad (7) \]

where

\[ \varepsilon_i = \varepsilon_i(t) = \begin{cases} 1, & \text{if } X_i(t) > \theta(t) \\ 0, & \text{if } X_i(t) \leq \theta(t) \end{cases} \]

and \( X_i(t) \) is the process generated by \( i \)-th ancestor, we have

\[ \lim_{t \to \infty} \ln E S_i^{y(x_2, t) - y(x_1, t)} = \lim_{t \to \infty} \left( [x_2 t] - [x_1 t] \right) R(\theta(t), t)(s - 1). \]

Thus again due to the limit theorem for critical processes we obtain (6) from the last relation.

It follows from (6) and (7) that

\[ \lim_{t \to \infty} E \left[ \prod_{i=1}^{r} S_i^{y(x_i, t) - y(x_{i-1}, t)} \right] = \exp \left\{ \sum_{i=1}^{r} \frac{2(x_i - x_{i-1})}{\sigma^2}(s_i - 1) \right\}. \]

Since the last limit is the generating function of \( (y(x_i) - y(x_{i-1}), i = 1, \ldots, r) \), we conclude that joint distributions of increments of \( y(x, t) \) tend to ones of \( y(x) \). According to Corollary 1 to Theorem 5 in Billingsley (1968) (see Billingsley (1968), p. 31) from convergence of increments we obtain that

\[ (y(x_1, t), \ldots, y(x_r, t)) \to (y(x_1), \ldots, y(x_r)), \]

as \( t \to \infty \) in distribution for any \( r \geq 1 \) and \( \theta \in [0, \infty) \).

Thus theorem is proved for \( \theta \in [0, \infty) \).

The proof for \( \theta = \infty \) follows from the fact that in this case the limit on the right side of (5) is zero.

**Proof of Theorem 2.** It follows from (2) and definition of the process \( W_t^{(i)}(x) \) that

\[ E e^{i\lambda W_t^{(i)}(x)} = \exp \left\{ -i\lambda n R(\theta(t), t) \frac{\sqrt{a(t)}}{\sqrt{2a(t)}} \right\} (1 - (1 - s)R(\theta(t), t))^{n}, \]

where \( n = [ta(t)x], s = e^{i\lambda / \sqrt{a(t)}} \). If we use the following Taylor expansions

\[ \ln(1 - x) = -x + O(x^2), \quad x \to 0 \quad (8) \]

\[ e^{ia} = 1 + ia - \frac{a^2}{2} + o(a^2), \quad a \to \infty, \quad (9) \]

we obtain

\[ \ln E e^{i\lambda W_t^{(i)}(x)} = -[ta(t)x] \frac{\lambda^2}{2a(t)} R(\theta(t), t) + o(tR(\theta(t), t)). \]

Taking into account relation (4) we conclude that

\[ \lim_{t \to \infty} E e^{i\lambda W_t^{(i)}(x)} = \exp \left\{ -\frac{\lambda^2 x}{\sigma^2} e^{-2\theta / \sigma^2} \right\}, \]

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which is the characteristic function (Fourier transform) of the one dimensional distribution of the Brownian motion $W^{(1)}(x)$.

Let $0 = x_0 < x_1 < \cdots < x_r < \infty$. To prove convergence of finite dimensional distributions we first show that the distribution of

$$ (W_t^{(1)}(x_j) - W_t^{(1)}(x_{j-1}), \quad j = 1, \ldots, r) $$

as $t \to \infty$ converges to the distribution of $(W^{(1)}(x_j) - W^{(1)}(x_{j-1}), \quad j = 1, \ldots, r)$.

Again using (2), definition of $W_t^{(1)}(x)$ and Taylor expansions (8), (9) we obtain that

$$ \lim_{t \to \infty} E \exp \left\{ i \sum_{j=1}^r \lambda_j [W_t^{(1)}(x_j) - W_t^{(1)}(x_{j-1})] \right\} $$

$$ = \exp \left\{ -\sigma^{-2} e^{-2\theta/\sigma^2} \sum_{j=1}^r \lambda_j^2 (x_j - x_{j-1}) \right\}, $$

for any $r \geq 1$, where $\lambda_j \in R, \quad j = 1, \ldots, r$. Since the last limit is the Fourier transform of the distribution of $(W^{(1)}(x_j) - W^{(1)}(x_{j-1}), j = 1, \ldots, r)$ we conclude from here that the joint distribution of increments of the process $W_t^{(1)}(x)$ converges as $t \to \infty$ to the joint distribution of Brownian motion’s increments.

Hence due to the mentioned above Corollary 1 in Billingsley (1968, p. 31) the finite dimensional distributions of $W_t^{(1)}(x)$ converges as $t \to \infty$ to ones of the Brownian motion $W^{(1)}(x)$ with zero shift and with diffusion parameter $2\sigma^{-1} \exp\{-2\theta/\sigma^2\}$. The theorem is proved for $\theta \in [0, \infty)$. The proof for $\theta = \infty$ follows from the same arguments, if we take into account that in this case the limit on the right side of (1) is 1. Theorem 2 is proved.

3. Supercritical processes

Now we consider the case of supercritical processes. It is known (Athreya, Ney (1972)) that if $A > 1$, $EX(1) \ln X(1) < \infty$, then $X(t)A^{-1}$ converges with probability one to a random variable $W$ and the Laplace transform $\varphi(\lambda)$ of $W$ satisfies the following equation

$$ \varphi(\lambda) = f \left( \varphi \left( \frac{\lambda}{A} \right) \right). $$

It is also known that the distribution function $\pi(x)$ of $W$ is absolute continuous for $x > 0$ and has an atom of the mass $q$ at $x = 0$. Here $q$ is the extinction probability.

We assume that there exists

$$ \lim_{t \to \infty} \theta(t)A^{-1} = \theta \in [0, \infty] $$

and

$$ \sum_{k=2}^{\infty} k P_k \ln k < \infty. $$

and consider “discrete time” process $S(n,t)$, $n = 0, 1, \ldots$ for $t \in N_0$. Note that here $n$ is the time parameter.
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Theorem 3. If \( A > 1 \) and conditions (11) and (12) are satisfied, then \( S(n_t, t) \xrightarrow{D} \xi(n) \) \( n \in \mathbb{N}_0 \) as \( t \to \infty \), where \( \xi(n) \) is a stochastic process with independent and binomially distributed increments such that

\[
P \{ \xi(n_i) - \xi(n_{i-1}) = k \} = \binom{n_i - n_{i-1}}{k} [1 - \pi(\theta)]^k \pi(\theta)^{n_i - n_{i-1} - k}
\]

for any \( 0 \leq n_{i-1} < n_i < \infty \), \( n_i \in \mathbb{N}_0 \), for \( \theta \in [0, \infty) \) and it is a zero process for \( \theta = \infty \).

Example 1. Let the offspring distribution be the positive geometric distribution, i.e. \( P_k = \alpha(1 - \alpha)^{k-1}, k \geq 1 \) and \( P_0 = 0 \). In this case the offspring generating function has the form \( f(s) = \alpha s(1 - \beta s)^{-1}, \beta = 1 - \alpha \) and \( A = \alpha^{-1} \) and the equation for the Laplace transform is:

\[
\varphi \left( \frac{\lambda}{\alpha} \right) = \frac{\alpha \varphi(\lambda)}{1 - \beta \varphi(\lambda)}.
\]

Now it is not difficult to check that the Laplace transform \( \varphi(\lambda) = \alpha(\alpha + \lambda)^{-1} \) satisfies the above equation. Hence the limit distribution \( \pi(x) \) is exponential with the density function \( \alpha e^{-\alpha x} \) and Theorem 3 gives the following result.

Corollary. If conditions of Theorem 3 are satisfied and the offspring distribution is the positive geometric of the parameter \( \alpha < \alpha < 1 \), then for \( \theta \in [0, \infty) \) the limit process \( \xi(n) \) in Theorem 3 is binomial such that

\[
P \{ \xi(n_i) - \xi(n_{i-1}) = k \} = \binom{n_i - n_{i-1}}{k} e^{-\alpha \theta k} [1 - e^{-\alpha \theta}]^{n_i - n_{i-1} - k}.
\]

Proof of Theorem 3. Let \( n_0, n_1, \ldots, n_r \) be such number that \( 0 = n_0 < n_1 < \cdots < n_r < \infty \) and \( n_i \in \mathbb{N}_0 \), \( 0 \leq i \leq r \). First we prove that for \( 1 \leq i \leq r \)

\[
\lim_{t \to \infty} ES^{S(n_i, t) - S(n_{i-1}, t)} = (\hat{\pi}(\theta) S_i + \pi(\theta))^{n_i - n_{i-1}}, \tag{13}
\]

where \( \hat{\pi}(\theta) = 1 - \pi(\theta) \). It follows from representation (7) that

\[
ES^{S(n_i, t) - S(n_{i-1}, t)} = (R(\theta(t), t) S_i + 1 - R(\theta(t), t))^{n_i - n_{i-1}}. \tag{14}
\]

Now we consider the estimate

\[
|P \{ X(t) \leq \theta(t) \} - \pi(\theta) | \leq \sup_x |P \{ X(t) A^{-1} \leq x \} - \pi(x) | + |\pi(\theta(t) A^{-1}) - \pi(\theta) |. \tag{15}
\]

First term on the right side of (15) tends to zero as \( t \to \infty \) due to the limit theorem for supercritical processes. It follows from condition (11) and continuity of \( \pi(x) \) that the limit of the second term is also zero. Thus

\[
R(\theta(t), t) \to 1 - \pi(\theta), \tag{16}
\]

as \( t \to \infty \). From relations (14) and (16) we obtain (13).
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Using independence of increments of $S(n, t)$ from relation (13) we have
\[
\lim_{t \to \infty} E \left[ \prod_{i=1}^{r} S^{(n_i, t)} - S^{(n_{i-1}, t)} \right] = \prod_{i=1}^{r} \left\{ \pi(\theta) S_i + \pi(\theta) \right\}^{n_i - n_{i-1}},
\]
which proves the theorem for $\theta \in [0, \infty)$. In the case $\theta = \infty$ the limit on the right side of (13) equals 1 and the "limit process" $\xi(n)$ equals zero for all $n$. The theorem is proved.

Theorem 3 shows that stochastic process $S(n, t)$ for fixed $n \in \mathbb{N}_0$ can be approximated as $t \to \infty$ by a binomial process. Now we consider the case when $n \to \infty$ together with $t$. Let $a(t)$ be a positive function such that $a(t) \to \infty$ as $t \to \infty$. We consider the following stochastic process
\[
W^{(2)}(x) = \frac{S([a(t)x], t) - [a(t)x]R(\theta(t), t)}{\sqrt{a(t)}},
\]
where $x \in [0, \infty)$.

**Theorem 4.** If $A > 1$ and conditions (11) and (12) are satisfied, then $W^{(2)}(x) \overset{D}{\to} W^{(2)}(x)$ as $t \to \infty$, where $W^{(2)}(x)$ is the Brownian motion with zero shift and with diffusion parameter $\pi(\theta)(1 - \pi(\theta))$ for $\theta \in [0, \infty)$ and it is a zero process for $\theta = \infty$.

**Example 2.** If, as in Example 1, the offspring distribution is the positive geometric of the parameter $0 < \alpha < 1$, then it is not difficult to see that the Brownian motion in Theorem 4 has the diffusion parameter $e^{-\alpha \theta}(1 - e^{-\alpha \theta})$.

**Proof.** First we prove the convergence of the one dimensional distribution. Let
\[
A(\lambda) = E e^{i\lambda W^{(2)}(x)}, B(\lambda) = \exp \left\{ -\frac{\lambda^2}{2a(t)} \sum_{j=1}^{[a(t)x]} \text{var} \varepsilon_j \right\},
\]
where $\varepsilon_j = \varepsilon_j(t), j = 1, 2, \ldots$ are the same as in the representation (7). Note that it follows from the definition of $W^{(2)}(x)$ and (7) that
\[
A(\lambda) = \prod_{j=1}^{[a(t)x]} E e^{i\lambda(\varepsilon_j - R(t))/\sqrt{a(t)}},
\]
where $R(t) = R(\theta(t), t)$.

Using inequality $\prod_j a_j - \prod_j b_j \leq \sum_j |a_j - b_j|, |a_j| \leq 1, |b_j| \leq 1$, and taking into account the fact that $E(\varepsilon_j - R(t)) = 0$ for $j = 1, 2, \ldots$, we have the following estimate
\[
|A(\lambda) - B(\lambda)| \leq T_1 + T_2,
\]
where with $\alpha_j(t) = \lambda^2 \text{var} (\varepsilon_j)/2a(t)$
\[
T_1 = \sum_{j=1}^{[a(t)x]} \left| e^{i\lambda(\varepsilon_j - R(t))/\sqrt{a(t)}} - 1 - \frac{\varepsilon_j - R(t)}{\sqrt{a(t)}} i\lambda + \alpha_j(t) \right|,
\]
and
\[
T_2 = \sum_{j=1}^{[a(t)x]} E \left| 1 - e^{-\alpha_j(t)} - \alpha_j(t) \right|.
\]

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Using the inequality

\[ |e^{i\alpha} - 1 - i\alpha + \frac{\alpha^2}{2}| \leq \frac{1}{3} |\alpha|^3, \]

we obtain that

\[ T_1 \leq \frac{[a(t)x]}{a^{3/2}(t)} E|\varepsilon_1 - R(t)|^3. \]

Here it is easy to see that \( E|\varepsilon_1 - R(t)|^3 \leq 2. \) Thus we have \( T_1 \to 0 \) as \( t \to \infty. \)

Taking into account that \( \text{var} \ v_j = \text{var} \ v_1, j = 1, \ldots \) is bounded and using the Taylor expansion \( e^{-x} = 1 - x + o(x), x \to 0, \) we obtain

\[ T_2 = [a(t)x]|\alpha_1(t) + o(\alpha_1(t)) - \alpha_1(t)| = o(1), \quad t \to \infty. \]

From these estimates and from (18) we conclude that functions \( A(\lambda) \) and \( B(\lambda) \) have the same limit as \( t \to \infty. \) On the other hand, since \( \text{var} \ v_j = R(\theta(t), t)(1 - R(\theta(t), t)), \)
the function \( B(\lambda) \) tends as \( t \to \infty \) to

\[ \exp \left\{ -\frac{\lambda^2}{2} x \pi(\theta)(1 - \pi(\theta)) \right\}, \]

which is the Fourier transform of the one dimensional distribution of the Brownian motion \( W^{(2)}(x). \)

Let \( 0 = x_0 < x_1 < \cdots < x_r < \infty, r \geq 1. \) It is not difficult to see that, if we repeat the above arguments, we obtain that the characteristic function \( E e^{i \lambda (W^{(2)}(x_j) - W^{(2)}(x_{j-1}))} \)
tends as \( t \to \infty \) to

\[ \exp \left\{ -\pi(\theta)(1 - \pi(\theta)) \frac{\lambda^2}{2} (x_j - x_{j-1}) \right\}, \]

for any \( j = 1, 2, \ldots, r \) and \( \theta \in [0, \infty). \) Therefore the limit of the characteristic function

\[ E \exp \left\{ i \sum_{j=1}^{k} \lambda_j \left[ W^{(2)}_t(x_j) - W^{(2)}_t(x_{j-1}) \right] \right\}, \]

as \( t \to \infty \) equals to

\[ \exp \left\{ -\pi(\theta)(1 - \pi(\theta)) \sum_{j=1}^{r} \frac{\lambda^2}{2} (x_j - x_{j-1}) \right\}. \]

Since the last limit is the Fourier transform of the joint distribution of increments of the Brownian motion \( W^{(2)}(x), \) we obtain convergence of finite dimensional distributions from the mentioned above (see proof of Theorem 1) Corollary 1 Billingsley (1968). Theorem 4 is proved.
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4. Subcritical processes

Let now $A < 1$, i.e., the initial process is subcritical. In this case we use the following limit theorem for subcritical processes (Sevastyanov (1971), p. 29). If $A < 1$, there exist

$$
\lim_{t \to \infty} P\{X(t) = j|X(t) > 0\} = P^*_j, \quad j \geq 1, \tag{19}
$$

and the generating function $F^*(s)$ of $P^*_j$, $j \geq 1$ satisfies the equation

$$
1 - F^*(s) = A(1 - F^*(s)). \tag{20}
$$

It is also known that, if $A \leq 1$, then $Q(t) = R(0, t) \to 0$ as $t \to \infty$. If $A < 1$ and in addition $EX(1) \ln X(1) < \infty$, then we have the following asymptotics for $Q(t)$ (see Sevastyanov (1971), p. 56)

$$
Q(t) \sim KA^t, 0 < K = \prod_{m=0}^{\infty} B(P\{X(m) = 0\}) < \infty, \tag{21}
$$

where $B(s) = (1 - f(s))/(A(1 - s))$.

Let $y(x, t) = S([x A^{-t}], t)$.

**Theorem 5.** If $A < 1$ and (12) is satisfied, then $y(x, t) \xrightarrow{D} y(x)$ as $t \to \infty$, where $y(x)$ is the Poisson process with $Ey(x) = Kx \sum_{j > \theta} P^*_j$ for $\theta(t) \equiv \theta \in \mathbb{N}_0$ and it is a zero process if $\theta(t) \to \infty$.

**Proof.** We use again (2) with $n = [x A^{-t}]$. In this case it follows from the above mentioned limit theorem for subcritical processes that under the condition (12)

$$
R(\theta(t), t) \sim KA^t \sum_{j > \theta} P^*_j, \quad t \to \infty, \tag{22}
$$

for $\theta \in \mathbb{N}_0$. Since

$$
\lim_{t \to \infty} \ln ES^{y(x, t)} = (s - 1) \lim_{t \to \infty} [x A^{-t}] R(\theta(t), t)
$$

we obtain that the limit of $ES^{y(x, t)}$ as $t \to \infty$ is $\exp\left\{Kx \sum_{j > \theta} P^*_j (s - 1)\right\}$.

To prove convergence of finite dimensional distribution it is sufficient to show that

$$
\lim_{t \to \infty} ES^{y(x_2, t) - y(x_1, t)} = \exp\left\{K(x_2 - x_1) \sum_{j > \theta} P^*_j (s - 1)\right\}, \tag{23}
$$

for any $0 < x_1 < x_2 < \infty$ and $0 < s < 1$. It follows from representation (7) and (2), that in this case

$$
\lim_{t \to \infty} \ln ES^{y(x_2, t) - y(x_1, t)} = \lim_{t \to \infty} n(t) R(\theta(t), t)(s - 1),
$$

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where \( n(t) = [x_2A^{-t}] - [x_1A^{-t}] \). We obtain (23) from here taking into account relation (22). Thus due to (23)

\[
\lim_{t \to \infty} E \prod_{j=1}^r S^{y(x_j,t)-y(x_{j-1},t)} = \exp \left\{ K(s-1) \sum_{i>\theta} P_i^* \sum_{j=1}^r (x_j - x_{j-1}) \right\},
\]

i.e., \((y(x_j,t) - y(x_{j-1},t), \quad i = 1, 2, \ldots r)\) converges in distribution to \((y(x_j) - y(x_{j-1}), \quad j = 1, 2, \ldots, r)\).

We obtain convergence of finite dimensional distributions from convergence of increments by the Corollary 1 in Billingsley (1968) (see Billingsley (1969, p. 31) as in the proof of previous theorems. Theorem 5 is proved.

Now we consider the case \( nA^t \to \infty \). Let, as before, \( a(t) \) be a positive valued function such that \( a(t) \to \infty \) as \( t \to \infty \). We define process \( W^{(3)}_t(x) \) by the relation

\[
W^{(3)}_t(x) = \frac{1}{\sqrt{a(t)}} \left\{ S^{|xA^{-t}a(t)|}, t - |xA^{-t}a(t)|R(\theta(t), t) \right\},
\]

where \( x \in [0, \infty) \).

**Theorem 6.** If \( A < 1 \) and (12) is satisfied, then \( W^{(3)}_t(x) \overset{D}{\to} W^{(3)}(x) \) as \( t \to \infty \), where \( W^{(3)}(x) \) is the Brownian motion with zero shift and with diffusion parameter \( K \sum_{i>\theta} P_i^* \) for \( \theta(t) = \theta \in \mathbb{N}_0 \) and it is a zero process if \( \theta(t) \to \infty \).

**Proof.** Let \( 0 \leq x_0 < x_1 < \infty \). It follows from definition of \( W^{(3)}_t(x) \) and representation (7) that

\[
A(\lambda) = Ee^{i\lambda(W^{(3)}_t(x_1) - W^{(3)}_t(x_0))} = \exp \left\{ -\frac{i\lambda n R(\theta(t), t)}{\sqrt{a(t)}} \right\} (1 - (1-s)R(\theta(t), t))^{n}, \quad (24)
\]

where \( n = |x_1A^{-t}a(t)| - |x_0A^{-t}a(t)| \) and \( S = e^{i\lambda/\sqrt{a(t)}} \). If we use Taylor expansions (8) an (9), we have

\[
\ln A(\lambda) = -\frac{\lambda^2}{2a(t)} n R(\theta, t) + o(A^{-t}R(\theta, t)), \quad t \to \infty.
\]

Taking into account relation (22) we obtain from here that

\[
A(\lambda) = \exp \left\{ -\frac{\lambda^2}{2} (x_1 - x_0) K \sum_{j>\theta} P_j^* \right\} + o(1), \quad t \to \infty. \quad (25)
\]

Let now \( 0 = x_0 < x_1 < \cdots < x_r < \infty, \quad r \geq 1 \). Since lives of different individuals are independent, the increments \( W^{(3)}_t(x_j) - W^{(3)}_t(x_{j-1}), \quad j = 1, \ldots, r, \) are independent. Therefore it follows from (25) that the joint distribution of these increments tends as \( t \to \infty \) to the joint distribution of increments \( W^{(3)}(x_j) - W^{(3)}(x_{j-1}), \quad j = 1, \ldots, r \). To obtain from here convergence of finite dimensional distributions we again appeal to the mentioned above Corollary 1 from Billingsley (1968). Theorem 6 is proved.
NUMBER OF PRODUCTIVE ANCESTORS

References


ÖZET

$n$ bireyden alınan bir kitlede bireylerin kesikli zamanlı dallanma süreçleri yarattığı düşünülmüş, atalar ve çocuklarının $t$ zamanı bakımından sayıları ele alınmıştır. Ataların sayısi olan $S(n, t)$'nin, $t$ sonsuza giderken artımları bağımsız olan bir süreçe yakınsadığı ispatlanmıştır.