THE INFLUENCE OF MOMENTS IN
BOOTSTRAP APPROXIMATION

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Abstract

The influence of moments on the rate of convergence of bootstrap distribution function, $F_n^*(x)$, of standardized arithmetic mean to its true distribution function $F_n(x)$ is studied in the case of i.i.d random variables $X_1, X_2, ..., X_n$ with $E(X) = 0$ and $E(X^2) < \infty$. It is verified that the existence of moment $E|X|^{2+\delta}$ is sufficient to have the rate of convergence to be $o(n^{-\delta/2})$ almost surely (a.s.) for $0 < \delta < 1$.

Key Words: Edgeword expansion, central limit theorem, characteristic function.

1. Introduction

Let $X_1, X_2, X_3, ..., X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.’s) with a symmetric distribution function (d.f.) $F(x)$ and $EX_i = 0$, $EX_i^2 = \sigma^2 < \infty$. Define, $T_n = \sum_{j=1}^{n} X_j$, $\bar{X}_n = n^{-1} T_n$, $S_n^2 = n^{-1} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2$. The distribution function of the standardized arithmetic mean will be denoted by $F_n(x) = P(T_n/\sqrt{n} \sigma \leq x)$. The specified r.v. of interest for the bootstrap is $T_n/\sqrt{n} \sigma$. Let $\hat{F}_n(x)$ denote empirical distribution function of $X_1, X_2, X_3, ..., X_n$, the distribution that puts mass $1/n$ at each point. The bootstrap is the name of a variety of resampling methods, namely, simple random sampling with replacement from the original sample. It is to approximate the d.f. of $T_n/\sqrt{n} \sigma$ by $T_n^*/\sqrt{n} S_n$ under $\hat{F}_n(x)$ where $T_n^* = n^{-1} \sum_{j=1}^{n} X_j^*$. Here, $X_1^*, X_2^*, X_3^*, ..., X_n^*$ is a random sample of size $n$ from $\hat{F}_n(x)$. We use notation $F_n^*(x) = P(T_n^*/\sqrt{n} S_n \leq x)$ for the bootstrap approximation of $F_n(x)$. The main concern of this paper is to find the conditions by means of the moments to have

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |F_n(x) - F_n^*(x)| < \infty \quad \text{a.s.}$$

where $0 < \delta < 1$. This requires finding some sort of lower bound for the rate of convergence of the bootstrap approximation $F_n^*(x)$ to the true distribution $F_n(x)$. The studies of this kind date back to 1960’s and its development can be found in Hall(1982). The rationale of considering such a summation for the rate of convergence of $\sup_x |F_n(x) - F_n^*(x)|$ can
be explained by the words of Baum and Katz (1965): One way of measuring the rate of convergence of non-negative and bounded sequence \( \{c_n\} \) is to determine \( r \geq -1 \) the series \( \sum_{n=1}^{\infty} n^r c_n \) converges if there is any. The idea of connecting the rate of convergence to the moments however, mainly due to Ibragimov (1966).

2. Background and Result

There are several results on the issue which are related with the result of this paper. Freidman, Katz and Koopmans (1966), Ibragimov (1966), Heyde (1967) gave connected results on the rate of convergence of \( F_n(x) \) to the normal distribution function \( \Phi(x) \). The following result belongs to Heyde (1967):

**Theorem 2.1.** Let \( X_i, i = 1, 2, 3, \ldots \) be a sequence of i.i.d. random variables with \( EX_i^2 = \sigma^2 < \infty \), \( EX_i = 0 \). Then

\[
\sum_{n=1}^{\infty} n^{-1+\delta} \sup_x |F_n(x) - \Phi(x)| < \infty, 0 \leq \delta < 1
\]

if and only if \( E|X_i|^{2+\delta} < \infty \), \( 0 < \delta < 1 \), \( EX_i^2 \ln(1 + |X_i|) < \infty \), \( \delta = 0 \).

This result has been extended for the difference between \( F_n(x) \) and a portion of its \((k+1)\) term Edgeworth expansion, \( G_{kn}(x) \), by Galtsyan (1971) and Heyde and Leslie (1972) independently of the other. This will be given below for the sake of completeness and to give some idea about the possible extensions:

**Theorem 2.2.** In order that

\[
\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) - G_{kn}| < \infty
\]

where \( k \) is a non-negative integer and \( 0 < \delta < 1 \), it is necessary and for \( k=0 \) or for distributions satisfying Cramer's condition also sufficient that \( E|X_j|^k \delta^{k+2} < \infty \).

The last theorem was established for the case \( \delta = 0 \) in both Galtsyan (1971), Heyde and Leslie (1972)

An another exploitation of the forementioned results is to set up them as theorems to find out the rate of convergence to zero of \( \sup_x |F_n(x) - F_n(x)| \); the rate of convergence of naive bootstrap approximation \( F_n^*(x) \) to the d.f. of the standardized arithmetic mean \( F_n(x) \). One result has been obtained by Hall (1988) (Theorem 3.1.i) in this direction. It is excerpted from Hall (1988):

**Theorem 2.3.** Let \( X_1, X_2, X_3, \ldots, X_n \) be a sequence of i.i.d. r.v.'s with d.f. \( F(x) \) and \( EX_i = 0 \), \( EX_i^2 = 1 \), and define the bootstrapped \( T_n \) as \( T_n^* = n^{-1/2} \sum_{i=1}^{n} X_i^* \) and its d.f. \( F_n^*(x) \). Then,

\[
\sup_{-\infty < x < \infty} |F_n^*(x) - \{\Phi(x) - n^{-1/2} \beta \phi(x^2 - 1)\}| = o(n^{-1/2}) \text{ a.s.}
\]

as \( n \to \infty \) if and only if \( E|X_i|^3 < \infty \) and \( EX_i^2 = \beta \).

What Hall is utilized as a technique to derive this result is called leading term approach and it is a bootstrap counterpart of Theorem 2.2. above for \( \delta = 0 \) and \( k = 1 \). Next, we will state the result of this work.
Theorem 2.4. Let $X_1, X_2, X_3, \ldots, X_n$ be a sequence of i.i.d. r.v.’s with a symmetric d.f. $F(x)$, $E X_j = 0$, $EX_j^2 = \sigma^2 < \infty$ and $\hat{F}_n(x)$ denote symmetrized empirical distribution function. Then,

$$\sum_{n=1}^{\infty} n^{-1+\frac{\delta}{2}} \sup_x |F_n^*(x) - F_n(x)| < \infty \text{ a.s.}$$

(2.1)

if $E|X_j|^{2+\delta} < \infty$, where $0 < \delta < 1$.

Remark 1. The symmetry assumption in Theorem 2.4. on the underlying distribution is not indispensable. It can be eliminated as it is done in Heyde(1971).

Remark 2. No further justification is needed for the existence of the second moment; because, it has been shown by Athreya(1987) that naive bootstrap fails unless $EX^2 < \infty$ in approximating the d.f. of the appropriately normalized $\bar{X}_n$.

The empirical d.f. to resample is symmetrized in order to comply with the underlying d.f. $F_n(x)$. The symmetrization can be achieved by following either Efron(1979)’s or Babu and Singh(1984)’s suggestions. According to the first suggestion the both positive and negative values of the differences $|X_j(w) - \bar{X}_n|$ are taken into the consideration in constructing symmetrical empirical d.f. $\hat{F}_n(x)$. In this case the symmetrical empirical d.f. is formed as the following:

$$\hat{F}_n(x) = \frac{1}{2n} \sum_{j=1}^{2n} I(X_j(w) - \bar{X}_n \leq x).$$

Babu and Singh’s method(1984) corrects the skewness as follows:

$$\hat{F}_n(x) = \frac{1}{2} [\hat{F}_n(x) + \hat{F}_n(2x - \bar{X}_n)].$$

The first approach will be preferred in the proof of Theorem 2.4. The result of this paper is a verification the bootstrap approximation $F_n^*(x)$ of $F_n(x)$ is as close as the approximation supplied by the normal theory by means of sufficient moment condition $E|X_j|^{2+\delta} < \infty$ for $0 < \delta < 1$.

3. Proof

It will be convenient to give the main ingredients of the proof before outlining it. Three lemmas will be stated below.

Lemma 3.1. Let $Q(t/\sqrt{n}\sigma)$ and $Q^*(t/\sqrt{n}s_n)$ be the characteristic functions of the r.v.’s $T_n/\sqrt{n}\sigma$ and its bootstrap counterpart respectively. If $E|X_j| < \infty$ then

$$\sup\{|Q^*(t/\sqrt{n}s_n) - Q(t/\sqrt{n}\sigma)| : |t| < \sqrt{n}\sigma M\} = o(1)$$

a.s. as $n \to \infty$ for a real number $M > 0$.

The proof of the lemma is made by combining the discretization method given by Babu and Singh(1984) and Lemma 4.2 of Lahiri(1989). This lemma is used for the proof of Lemma 3.2 in the sequel, and given as:

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**Lemma 3.2.** Let $X_1, X_2, ..., X_n$, be a sequence of i.i.d. random variables with a symmetric d.f. $F(x)$ and $E|X_1|^r < \infty$ for a real number $r \geq 2$. Let $\kappa_s^*$ denote $s$th order cumulants of the $X_s^*$ with the symmetrized empirical d.f. $F_n^*(x)$. The characteristic function of r.v. $X_s^* - \bar{X}_n$ can be written as

$$Q^*(t) = e^{\sum_{s=2}^{r} \frac{(it)^s}{s!}\kappa_s^* + |t|^r \gamma^*(t)} \text{ a.s.}$$

as $n \to \infty$ Here, $\gamma^*(t) = o(1)$ a.s. as $t \to 0$, $n \to \infty$ and there exist $\epsilon > 0$ small enough for which $|\gamma^*(t)| \neq 0$ a.s. $n \to \infty$. Under the same assumptions, the characteristic function can be represented as

$$Q^*(t) = \sum_{s=1}^{r} \frac{(it)^s}{s!}\mu_s^* + |t|^r \beta^*(t)$$

a.s. as $n \to \infty$ where $\mu_s^*$ is the $s$th order expected value of the r.v. $X_s^* - \bar{X}_n$ and similarly $|\beta^*(t)| = o(1)$ a.s. as $t \to 0$ and $n \to \infty$.

The detailed proof of the lemma can be found in Karabulut(1995). It is a modified form of Theorem 1.6.1. of Ibragimov(1966) or Heyde and Leslie(1972).

**Lemma 3.3.** If $S_n^2 \to \sigma^2$ and $0 < u \leq \sqrt{n}S_n$ then

$$|e^{-\frac{1}{2}u^2nS_n^2} - e^{-\frac{1}{2}u^2n\sigma^2}| = o(1) \text{ a.s.}$$

It is a simple result of Proposition 1.2.16. of Rao(1987).

Now we are ready to give the outline of the proof of Theorem 2.4. of this paper. It will follow the line that of Heyde (1967). That method is based on the development made by Ibragimov(1967). First, note that

$$\sup_x |F_n(x) - F_n^*(x)| \leq \sup_x |F_n(x) - \Phi(x)| + \sup_x |F_n^*(x) - \Phi(x)|.$$  

The result related for the first term on the right hand side provided by Heyde(1967), so it is enough to consider the results related with the second term on the right hand site. The outline of the proof begins with the verification of the following lemma.

**Lemma 3.4.** Let $X_1, X_2, ..., X_n$ be a sequence of r.v.'s as in Theorem 2.4. and $\gamma^*(t)$ as defined in Lemma 3.2 The series in (2.1) converges if and only if

$$\int_0^A \frac{\gamma^*(u)}{u^{1+\delta}} du < \infty$$  \hspace{1cm} (3.2)

a.s. as $n \to \infty$ for a real $A > 0$ and $0 < \delta < 1$.

Using this lemma, it will be shown that (3.2) is equivalent to the existence of the moment condition of Theorem 2.4.

First assume that the series in (2.1.) is convergent, i.e.

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |F_n^*(x) - \Phi(x)| < \infty \text{ a.s.}$$  \hspace{1cm} (3.3)
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Now, it will be shown that (3.3) and

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \int_{0}^{1} |Q_n^*\left(\frac{t}{\sqrt{nS_n}}\right) - e^{-t^2/2}| e^{-t^2/2} dt < \infty$$

(3.4)
a.s. are equivalent. By employing Parseval relation (see Feller (1971)) to (3.4) one can have

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \int_{-\infty}^{\infty} \left|Q_n^*\left(\frac{t}{\sqrt{nS_n}}\right) - e^{-t^2/2}\right| e^{-t^2/2} dt$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_{x} \left|F_n^*(x) - \Phi(x)\right|$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\delta/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(F_n^*(x) - \Phi(x)\right) x e^{-x^2/2} dx$$

When Mill ratio is used the equivalence of (3.3) and (3.4) follows. Because of the symmetry assumption in Lemma 3.2, the sign of the integrand in (3.4) does not change in intervals $(0, \epsilon)$ or $(-\epsilon, 0)$, and thus in the interval $0 < t/\sqrt{nS_n} < \epsilon$. If some manipulations is made on the integrand of (3.4) to be able to use the inequality

$$|1 - e^x| \leq |x| e^{|x|}$$

and using the fact that we can find a constant (possibly random) $c > 0$ such that

$$\max_{0 \leq t/\sqrt{nS_n} \leq c} \left|\gamma^*\left(\frac{t}{\sqrt{nS_n}}\right)\right| \leq 1/2 \text{ a.s.}$$

(3.5)
as $n \to \infty$. Using (3.5) and the transformation $u = t/\sqrt{nS_n}$, (3.4) becomes

$$\sum_{n=1}^{\infty} n^{1/2(1+\delta)} \int_{0}^{1/\sqrt{nS_n}} S_n^{u^2/2} |\gamma^*(u)| du < \infty \text{ a.s.}$$

(3.6)

Since the summand in (3.6) is monotonically decreasing, by Lemma 3.2 $|\gamma^*(u)| \to 0$ a.s. as $u \to 0$ in addition to decreasing upper limit of the integral. The summation in (3.6) can be changed to integration by the integral test theorem as in Apostol (1974). To do so $n$ is replaced by $x > 1$. Let $[x]$ denote integer valued function and remember that $[x] \leq x < 1 + [x]$. Also it is possible to find a constant $c > 0$ such that $(n + 1)^{1/2(1+\delta)} < cn^{1/2(1+\delta)}$

Therefore, by making use of these results it is seen that

$$\int_{0}^{X} x^{1/2(1+\delta)} \left(\int_{0}^{1/\sqrt{x}S_n} u^2 |\gamma^*(u)| du\right) dx$$

$$\leq \sum_{n=1}^{[X]} \int_{n}^{n+1} x^{1/2(1+\delta)} \left(\int_{0}^{1/\sqrt{x}S_n} u^2 |\gamma^*(u)| du\right) dx$$

$$< \sum_{n=1}^{[X]} n^{1/2(1+\delta)} \int_{0}^{1/\sqrt{nS_n}} u^2 |\gamma^*(u)| du$$

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and
\[ \int_1^{\infty} x^{\frac{1}{2} (1+\delta)} \left( \int_0^{1/\sqrt{ES_n}} u^2 |\gamma^*(u)| du \right) dx < \infty \text{ a.s.} \]

When \( w \) tends to infinity we should have
\[ \int_w^{2w} x^{\frac{1}{2} (1+\delta)} \left( \int_0^{1/\sqrt{ES_n}} u^2 |\gamma^*(u)| du \right) dx \to 0 \text{ a.s.} \] \hspace{1cm} (3.7)

The integration in the parenthesis is monotonically decreasing. Therefore, the transformation \( v = \frac{1}{\sqrt{2wES_n}} \) after \( x \) is replaced by \( 2w \) in the upper limit of the integral gives that
\[ \frac{1}{v^{(3+\delta)}} \int_0^{v} u^2 |\gamma^*(u)| du \to 0 \]
as \( v \to 0 \). If the partial integration is performed to (3.7) after making transformation \( v = \frac{1}{\sqrt{ES_n}} \) and taking \( A = S_n^{-1} \) at the end we obtain (3.2).

For the sufficiency we will begin with the use of Berry- Esseeén Lemma 16.3.2 of Feller(1971). To do so first, take the integral limit as \( T = B \sqrt{nS_n} \) with the constant \( B > 0 \) and make use of Lemma 3.2 again to choose a constant \( c \), (possibly random) satisfying \( 0 < c < B \). Hence,
\[ |Q_n^*(t) - e^{-t^2/2}| \leq \frac{t^2}{2} |\gamma^*\left(\frac{t}{\sqrt{nS_n}}\right)| \text{ a.s.} \] \hspace{1cm} (3.8)

The second term on the right hand side of Berry - Esseeén inequality can be neglected because of \( o(\frac{1}{n^{1/2}}) \) a.s.. Replacing (3.8) in (2.1) by using the symmetry property of (3.8) and omitting all the constants we find the inequality
\[ \sum_{n=1}^{\infty} n^{-1+\frac{1}{2}} \int_{-T}^{T} \frac{Q_n^*(t) - e^{-t^2/2}}{t} dt \leq \sum_{n=1}^{\infty} n^{-1+\frac{1}{2}} \int_0^{c\sqrt{nS_n}} t |\gamma^*\left(\frac{t}{\sqrt{nS_n}}\right)| dt. \]

By changing variable, \( u = \frac{t}{\sqrt{nS_n}} \) and changing the order of summation and integration, the positiveness of all terms permits this, the right hand side of the last inequality becomes
\[ \sum_{n=1}^{\infty} n^{-1+\frac{1}{2}} \int_{-T}^{T} \left| Q_n^*(t) - e^{-t^2/2} \right| dt \leq S_n^2 \int_0^{c} u |\gamma^*(u)| \left[ \sum_{n=1}^{\infty} n^{\frac{1}{2}} u^{-\frac{1}{2}} nS_n^2 \right] du \]

An Abelian theorem in Apostol(1974) will be utilized with the random terms to simplify the right hand side of the last inequality above. Now, Lemma 3.3. is invoked to make sure that the random sum is finite. Combining all the forementioned results together with the fact that \( 1 - e^{-x} > x - x^2/2 \) and \( 1/4S_n^2 u^2 < 1 \) for \( n \) large enough we can find a positive constant \( c \) such that
\[ \sum_{n=1}^{\infty} n^{-1+\frac{1}{2}} \int_{-T}^{T} \left| Q_n^*(t) - e^{-t^2/2} \right| dt \leq c \int_0^{c} \frac{\left| \gamma^*(u) \right|}{u^{(1+\delta)}(1-\frac{1}{8}S_n^2 u^2)^{(1+\frac{1}{2})}} du \]
The result follows after taking partial integration of the right hand side of the inequality and using (3.2).

The next part of the proof will be completed by showing that (3.2) is equivalent to the existence of the moment with the stated order in Theorem 2.4.

By replacing \((t^2/2)|\gamma^*(t)|\) in (3.2) with its equivalence in Lemma 3.2 and remembering that \(\tilde{F}_n(x)\) is symmetric, the following calculation is possible. Assume that \(\gamma^*(t) < 0\) without losing any generality and make use of that \((\cos x - 1 + x^2/2) \geq 0\) (see Ibragimov(1967) or Hall(1982)) to reach the following

\[
\frac{S_n^2}{2} \int_0^\infty |\gamma^*(t)| \frac{dt}{t(1+\delta)} = \int_0^\infty \frac{\int_{-\infty}^\infty \left(e^{\frac{t(X_j - \bar{X}_n)}{S_n}} - 1 + \frac{t^2 (X_j - \bar{X}_n)^2}{2 S_n^2}\right) dt}{t(1+\delta)} d\tilde{F}_n(x)
\]

\[
= \int_0^\infty \int_0^\infty \frac{\cos \frac{t(X_j - \bar{X}_n)}{S_n} - 1 + \frac{t^2 (X_j - \bar{X}_n)^2}{2 S_n^2}}{t(1+\delta)} dt d\tilde{F}_n(x)
\]

\[
< \infty
\]

a.s. as \(n \to \infty\). After applying partial integration two times to the integral on the right hand side

\[
\frac{1}{n} \sum_{j=1}^n \left[ \left(-\frac{\cos \frac{c(X_j - \bar{X}_n)}{S_n} - 1 + \frac{c^2 (X_j - \bar{X}_n)^2}{2 S_n^2}}{(2+\delta)c^{(2+\delta)}} + \left(\frac{-\frac{(X_j - \bar{X}_n)}{S_n} \sin \frac{c(X_j - \bar{X}_n)}{S_n} + \frac{c(X_j - \bar{X}_n)^2}{S_n^2}}{(2+\delta)c^{(2+\delta)}} + \left(\int_0^c \frac{(X_j - \bar{X}_n)^2 (1 - \cos \frac{t(X_j - \bar{X}_n)}{S_n})}{(2+\delta)c^{(2+\delta)}} dt \right) \right) \right]
\]

is obtained. The verification of the existence and finiteness of integral placed on the most right will give the necessary and sufficient condition which we seek. By making transformation \(v = t(X_j - \bar{X}_n)/S_n\) we get

\[
\frac{1}{n} \sum_{j=1}^n |X_j - \bar{X}_n|^{(2+\delta)} \int_0^c |X_j - \bar{X}_n|/S_n \frac{1 - \cos v}{v^{(1+\delta)}} dv
\]

The integral is finite and it can be neglected. Hence at the end, the integral in (3.2) is finite a.s. if the summation is finite a.s. It is to be finite we should have

\[
\frac{1}{n} \sum_{j=1}^n |X_j - \bar{X}_n|^{(2+\delta)} \rightarrow E|X_j|^{(2+\delta)} < \infty
\]

by the strong law of large numbers. Thus, it is true that \(E|X_j|^{(2+\delta)} < \infty\) is sufficient to have (2.1).
INFLUENCE OF MOMENTS

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References


Özet

Birbirinden bağımsız ve aynı dağılımlı ve $EX = 0$, $EX^2 < \infty$ olan $X_1, X_2, ..., X_n$ rassal değişkenlerinin aritmetik ortalama- malarına ait bootstrap dağılım fonksiyonu $F_n^*(x)$ın yine bu rassal değişkenin dağılımı $F_n(x)$'e yakınsamasında beklenen değerlerin etkisi araştırıldı. Yakınsama hızının hemen hemen heryerde $o(n^{-\delta/2})$ olabildiği için $0 < \delta < 1$ olmak üzere $E|X|^{2+\delta} < \infty$ olmasının yeterli olduğu gösterildi.