THE DISCRETE FOURIER TRANSFORM APPROXIMATION FOR PERIODICALLY CORRELATED TIME SERIES

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Abstract

The discrete Fourier transform is used in many statistical inference problems. An approximation of the discrete Fourier transform becomes very important especially in approximating the spectral density function and deriving some distributional properties. In this study, it is shown that a similar approximation derived for stationary time series is also valid for periodically correlated series.

Key Words: Discrete Fourier Transform, Spectral Density Function

1. Introduction

There are occasions when a pair of real-valued time series can most generally be regarded as a single complex-valued series. In general we are faced with strictly real-valued data. These can always be regarded as complex numbers with zero imaginary parts, although this may seem an unnecessary technicality. However, certain algebraic simplifications that arise make the required stretch of the imagination interestingly appealing.

Many problems of statistical inference for time series are based on the frequency-domain properties of the series. Spectral analysis for time series and in particular the estimation of the spectral density function, depends heavily on the asymptotic distribution as \( n \to \infty \) of the periodogram ordinates which is defined in terms of the discrete Fourier transform of the series \( \{X_0, X_1, ..., X_{n-1}\} \). Given a set of observations \( \{X_0, X_1, ..., X_{n-1}\} \) the discrete Fourier transform of the series is defined by

\[
\hat{X}(f) = \frac{1}{n} \sum_{t=0}^{n-1} X_t e^{-2\pi if t}
\] (1)
where \( f = 0, 1/n, 2/n, \ldots, (n - 1)/n \) (known as fundamental frequencies or Fourier frequencies and) \( i = \sqrt{-1} \).

In this paper, we study the asymptotic behavior of \( d_X(f) \) for periodically correlated time series with a known period \( \tau \). A stochastic process \( \{X_t : t \in T\} \) defined on a probability space \((\Omega, \mathcal{F}, P)\) and having finite second moments for all \( t \) is called second order stationary (sometimes covariance stationary, or simply stationary), if the following conditions hold:

1. \( E(X_t) = \mu \) for all \( t \) (that is, the mean is constant in time)
2. \( \gamma(h) = Cov(X_{t}, X_{t+r}) \) for all \( t \)

and the process is called periodically correlated (PC) with period \( \tau \), if

1a. \( E(X_t) = E(X_{t+r}) \) for all \( t \)
2a. \( Cov(X_{t}, X_{s}) = Cov(X_{t+r}, X_{t+r+s}) \) for all \( t \) and \( s \).

Without any loss of generality, we can assume \( E(X_t) = 0 \). Then (2a) reduces to \( E(X_{t}X_{t}^*) = E(X_{t+r}X_{t+r}^*) \). Here \( X^* \) denotes the complex conjugate of \( X \). In the above definition, if \( T \) is taken to be integers, then the process is called periodically correlated random sequence, and if \( T \) is taken to be all the real numbers, then \( \{X_t : t \in T\} \) is called continuous time periodically correlated stochastic process.

The periodically correlated processes are also known as periodically non stationary processes, cyclostationary, or sometimes periodically stationary time series. Some of the properties of the PC processes have been discussed extensively by many authors. Gladyshev (1961) gives the necessary and sufficient conditions for a function to be a correlation function of some periodically correlated series and provides a representation of a PC process in terms of a stationary time series. This is a very useful representation between the PC processes and stationary time series and this will be the key result in our study. Hurd (1989) gives a representation of a continuous time strongly harmonizable process and their covariances. Troutman (1979) considers a representation of a PC process as an infinite linear combination of independent, periodically distributed random variables. Jones and Brevisford (1967) give a method of prediction of time series with periodic structure. Bloomfield, Hurd and Lund (1992) look at the periodic correlation in the stratospheric ozone data. Brockwell and Davis (1987) give a good summary of the discrete Fourier transform of a time series in the context of Hilbert spaces and present some of its consistency properties. For a stationary time series a powerful identity is used in many statistical inference problems. Now, we will consider a stationary time series and review some of the properties of the identity mentioned above.

For an illustration assume that the set of random variables \( \{X_0, X_1, \ldots, X_{n-1}\} \) come from a stationary first order auto regressive time series \( X_t = \varphi X_{t-1} + \varepsilon_t \) with \( |\varphi| < 1 \) and that \( \varepsilon_t \)'s are independent and identically distributed random variables with mean 0 and variance \( \sigma^2 \). When we write the discrete Fourier transform for this series

\[
d_X(f) = \frac{1}{n} \sum_{t=0}^{n-1} X_t e^{-2\pi ift} = \frac{1}{n} \sum_{t=0}^{n-1} (\varphi X_{t-1} + \varepsilon_t) e^{-2\pi ift}
\]
THE DISCRETE FOURIER TRANSFORM

\[ = \varphi \sum_{t=0}^{n-1} X_{t-1} e^{-2\pi i ft} + d_c(f) = \varphi e^{-2\pi if} d_X(f) + d_c(f) - \varphi X_{n-1} e^{-2\pi ifn} \]

This implies that when \( f \) is taken to be \( 0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n} \) (known as Fourier frequencies or harmonic frequencies) and using the Euler’s identity \( e^{ix} = \cos(x) + i \sin(x) \) we have

\[ (1 - \varphi e^{-2\pi if}) d_X(f) = d_c(f) - \frac{1}{n} X_{n-1} \]

and

\[ \sqrt{n} d_X(f) = \sqrt{n} \frac{1}{1 - \varphi e^{-2\pi if}} d_c(f) - \sqrt{n} \frac{1}{n} \frac{1}{1 - \varphi e^{-2\pi if}} X_{n-1} \]

This can be written as

\[ \sqrt{n} d_X(f) = \sqrt{n} a(f) d_c(f) + R(n, f) \]

where \( a(f) = \frac{1}{1 - \varphi e^{-2\pi if}} \) and \( R(n, f) = -\frac{1}{n} \frac{1}{1 - \varphi e^{-2\pi if}} X_{n-1} \). Using the stationarity of \( X_t \), we can show that \( R(n, f) \overset{P}{\rightarrow} 0 \) as \( n \to \infty \) and \( \sqrt{n} a(f) d_c(f) = O_P(1) \).

It can easily be shown that for any stationary process we have the following identity holds:

\[ \sqrt{n} d_X(f) = \sqrt{n} a(f) d_c(f) + R(n, f) \quad (2) \]

where \( R(n, f) \overset{P}{\rightarrow} 0 \) as \( n \to \infty \). The periodogram ordinates of the series is defined in terms of the discrete Fourier transform as \( I(f) = n |d_X(f)|^2 \) and the spectral density function is the expected value of the periodogram ordinates. When \( \epsilon_t \) is a sequence of independent and identically distributed random variables with mean zero and variance \( \sigma^2 \), we have \( E(n |d_c(f)|^2) = \sigma^2 \) and the spectral density function of the series \( \{X_t : t = 0, 1, 2, ..., n-1\} \) can be approximated by \( S(f) = \sigma^2 |a(f)|^2 \). The spectral density function of stationary first order auto regressive time series can be seen to be \( S(f) = \frac{\sigma^2}{1 + \varphi^2 + 2\varphi \cos(2\pi f)} \). Thus the discrete Fourier transform of the series \( \{X_t : t = 0, 1, 2, ..., n-1\} \) can be written as \( d_X(f) \cong \sqrt{S(f)} d_c(f) \) where \( d_c(f) \) is the discrete Fourier transform of the white noise sequence and the periodogram ordinate of the series is approximated as \( I_X(f) \cong S_X(f) I_c(f) \). Here \( S(f) \) is the spectral density function of the series \( \{X_t : t = 0, 1, 2, ..., n-1\} \) which is sometimes written as \( S(f) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i fh} \) where \( \gamma(h) = Cov(X_t, X_{t+h}) \) is the autocovariance function of the series. Our purpose is to derive an identity similar to (2) for periodically correlated time series.
2. Periodically Correlated Series

As already mentioned, there is a useful representation between the periodically correlated series and stationary series. Gladyshev (1961) gives such a representation as

\[ X_t = \sum_{k=0}^{\tau-1} z_t^k e^{2\pi i k t / \tau} \]  

(3)

where \( z_t^k \) is the kth component of a \( \tau \)-dimensional stationary (in the wide sense) vector process \( Z_t = (z_t^0, z_t^1, ..., z_t^{\tau-1})^T \). Obviously, if a vector process is known to be stationary, then each component is marginally stationary. Therefore, for each \( k \), \( (k = 0, 1, 2, ..., \tau - 1) \) \( z_t^k \) is a univariate stationary time series and we can apply (2) to \( z_t^k \)

\[ \sqrt{n}d_{z^k}(f) = \sqrt{n}a_k(f)d_k(f) + R_k(n, f) \]  

(4)

where \( R_k(f, n) \to 0 \) as \( n \to \infty \). Using the equations (1) and (3), we obtain

\[ d_X(f) = \frac{1}{n} \sum_{t=0}^{n-1} X_t e^{-2\pi i ft} = \frac{1}{n} \sum_{t=0}^{n-1} \sum_{k=0}^{\tau-1} z_t^k e^{2\pi i k t / \tau} e^{-2\pi i ft} \]

\[ d_X(f) = \frac{1}{n} \sum_{k=0}^{\tau-1} \sum_{t=0}^{n-1} z_t^k e^{-2\pi i (f - k / \tau) t} = \sum_{k=0}^{\tau-1} d_{z^k}(f - k / \tau) \]  

(5)

for each \( k \) \( (k = 0, 1, 2, ..., \tau - 1) \), \( d_{z^k}(f - k / \tau) \) satisfies the identity (2), and thus from (5) we get

\[ \sqrt{n} d_X(f) = \sqrt{n} \sum_{k=0}^{\tau-1} d_{z^k}(f - k / \tau) \]

\[ = \sqrt{n} \sum_{k=0}^{\tau-1} a(f - k / \tau) d_{z^k}(f - k / \tau) + \sum_{k=0}^{\tau-1} R_k(n, f - k / \tau) \]  

(6)

In the final equation \( R_k(n, f - k / \tau) \to 0 \) as \( n \to \infty \) without depending on \( k \). Since the summand ranges from 0 to \( \tau - 1 \), the summation will go to zero in probability because the period \( \tau \) does not depend on \( n \). Thus we can give the following theorem which give the identity we seek:

**Theorem:** If the time series \( \{X_t : t = 0, 1, 2, ..., n - 1\} \) is periodically correlated time series with a known period \( \tau \), then

\[ \sqrt{n} d_X(f) = \sqrt{n} \sum_{k=0}^{\tau-1} a(f - k / \tau) d_{z^k}(f - k / \tau) + \sum_{k=0}^{\tau-1} R_k(f - k / \tau) \]  

(7)

where \( e^k \) is a white noise sequence corresponding to the kth component of the \( \tau \)-dimensional stationary vector time series, and \( \sum_{k=0}^{\tau-1} R_k(f - k / \tau) \to 0 \) as \( n \to \infty \).
THE DISCRETE FOURIER TRANSFORM

3. Examples

1. Assume that a set of random variables \( \{Y_0, Y_1, \ldots, Y_{n-1}\} \) form a first order moving average time series \( Y_t = \theta e_{t-1} + \epsilon_t \), where \( \epsilon_t \) is a sequence of independent and normally distributed random variables with mean zero and variance \( \sigma^2 \). The autocovariance function of this moving-average series is

\[
\gamma_Y(h) = \begin{cases} 
\sigma^2(1 + \theta)^2 & h = 0 \\
\sigma^2 \theta & h = \pm 1 \\
0 & \text{elsewhere}
\end{cases}
\]

The spectral density function can be calculated as

\[
S_Y(f) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi ifh} = \sigma^2(\theta e^{2\pi if} + 1 + \theta^2 + \theta e^{-2\pi if})
\]

\[
= \sigma^2 \left( 1 + \theta^2 + 2\theta \frac{e^{2\pi if} + e^{-2\pi if}}{2} \right) = \sigma^2(1 + \theta^2 + 2\theta \cos(2\pi f))
\]

We can find the spectral density function by using the discrete Fourier transform of the series. The discrete Fourier transform is given by

\[
d_Y(f) = \frac{1}{n} \sum_{t=0}^{n-1} Y_t e^{-2\pi ift} = \frac{1}{n} \sum_{t=0}^{n-1} (\epsilon_t + \theta \epsilon_{t-1}) e^{-2\pi ift} = d_Y(f) + \frac{\theta}{n} \sum_{t=0}^{n-1} \epsilon_{t-1} e^{-2\pi ift}
\]

\[
= d_e(f) + \theta e^{-2\pi if} \frac{1}{n} \sum_{t=0}^{n-1} \epsilon_t e^{-2\pi ift} - \frac{\theta}{n} \epsilon_{n-1} = (1 + \theta e^{-2\pi if}) d_e(f) + R(n, f)
\]

where \( R(n, f) \overset{P}{\to} 0 \) as \( n \to \infty \) and \( \theta(f) = 1 + \theta e^{-2\pi if} \). It is easy to see that \( \sqrt{n}d_e(f) = O_P(1) \). Defining the periodogram ordinate in terms of the discrete Fourier transform \( I_Y(f) = n |d_Y(f)|^2 \) and using the fact that \( E(I_Y(f)) = \sigma^2 \) we can write the spectral density function of the series as follows:

\[
S(f) = \sigma^2 |\theta(f)|^2 = \sigma^2(1 + \theta e^{2\pi if})(1 + \theta e^{-2\pi if}) = \sigma^2(1 + \theta^2 + 2\theta \cos(2\pi f))
\]

which is the same as before.

2. Now we consider the first order periodically correlated moving average series with period 2. The series \( Y_t = \epsilon_t + \theta \epsilon_{t-1} \) satisfies the conditions that the series to be periodically correlated, since mean is constant and

\[
Cov(Y_{t+\tau}, Y_{s+\tau}) = Cov(\epsilon_{t+\tau} + \theta \epsilon_{t+\tau-1}, \epsilon_{s+\tau} + \theta \epsilon_{s+\tau-1})
\]

51
From (7), we can write

$$\sqrt{n} d_Y(f) = \sqrt{n} \sum_{k=0}^{1} \theta(f - k/2) d_k(f - k/2) + \sum_{k=0}^{1} R_k(f - k/2)$$

and the periodogram ordinate is

$$I_Y(f) = n|d_Y(f)|^2 = n|\sum_{k=0}^{1} \theta(f - k/2) d_k(f - k/2)|^2$$  \hspace{1cm} (8)

If $\epsilon_t^0$ and $\epsilon_t^1$ independently distributed random variables then the identity (8) implies that the spectral density function can be written as $S_Y(f) = 2\sigma^2(1 + \theta^2)$. Now, assume that the vectors

$$\epsilon_t = \begin{pmatrix} \epsilon_t^0 \\ \epsilon_t^1 \end{pmatrix} \text{ are i.i.d with } \begin{pmatrix} 0 \\ \sigma^2 0 \rho \\ \rho \sigma^2 \end{pmatrix}$$

under the circumstance the spectral density function of the series can be calculated as follows:

$$S_Y(f) = E\{n|d_Y(f)|^2\} = nE\left\{\sum_{k=0}^{1} \theta(f - k/2) d_k(f - k/2)|^2\right\}$$

$$= nE\{(\theta(f) d_0(f) + \theta(f - 1/2) d_1(f - 1/2))^2 (\theta(f) d_0(f) + \theta(f - 1/2) d_1(f - 1/2))\}$$

$$= |\theta(f)|^2 E\{n|d_0(f)|^2\} + |\theta(f - 1/2)|^2 E\{n|d_1(f - 1/2)|^2\}$$

$$+ n\theta^*(f) \theta(f - 1/2) E\{d_0(f) d_1(f - 1/2)\} + n\theta^*(f - 1/2) \theta(f) E\{d_1(f - 1/2) d_0(f)\}$$

From the Euler's formula $a(f - 1/2) = 1 - \theta e^{-2\pi if}$ and $d_1(f - 1/2) = \frac{1}{n} \sum_{t=0}^{n-1} (-1)^t \epsilon_t e^{-2\pi it}$

Since

$$E\{d_0^*(f) d_1(f - 1/2)\} = E\left\{\frac{1}{n} \sum_{t=0}^{n-1} \epsilon_t^0 e^{2\pi it} \frac{1}{n} \sum_{s=0}^{n-1} \epsilon_s^1 e^{-2\pi is(f - 1/2)}\right\}$$

$$= \frac{1}{n^2} E\left\{\sum_{t=0}^{n-1} \epsilon_t^0 e^{2\pi it} \sum_{s=0}^{n-1} \epsilon_s^1 (-1)^s e^{-2\pi isf}\right\} = \frac{1}{n^2} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} (-1)^t e^{2\pi if(t-s)} E\{\epsilon_t^0 \epsilon_s^1\}$$

$$= \frac{\rho}{n^2} \left(\sum_{i=0}^{n-1} e^{2\pi if}\right) \left(\sum_{s=0}^{n-1} (-1)^s e^{-2\pi ifs}\right) = 0, \text{ because } \sum_{i=0}^{n-1} e^{2\pi if} = 0 \text{ for } f = k/n.$$
THE DISCRETE FOURIER TRANSFORM

\[ S(f) = |\theta(f)|^2 E\{n|d_{\theta}(f)|^2\} + |\theta(f - 1/2)|^2 E\{n|d_{\theta}(f - 1/2)|^2\} = 2\sigma^2(1 + \theta^2) \]

Since

\[ |a(f - 1/2)|^2 = |1 - \theta e^{-2\pi if}|^2 = (1 - \theta e^{2\pi if})(1 - \theta e^{-2\pi if}) = 1 + \theta^2 - 2\theta \cos(2\pi f) \]

and

\[ |a(f)|^2 = |1 + \theta e^{-2\pi if}|^2 = (1 + \theta e^{2\pi if})(1 + \theta e^{-2\pi if}) = 1 + \theta^2 + 2\theta \cos(2\pi f). \]

Hence, the spectral density function is

\[ S_Y(f) = 2\sigma^2(1 + \theta^2). \]

Conclusion

In this paper, the discrete Fourier transform approximation is applied to the periodically correlated time series. For an illustration, periodically correlated first order moving-average series is discussed.

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References


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