ON DUALITY BETWEEN CONTROL AND ESTIMATION PROBLEMS

Agamirza Bashirov, Huseyin Etikan and Nidai Semi
Department of Mathematics
Eastern Mediterranean University
Gazimagusa - North Cyprus

Abstract
In this paper, the dual control problems for the linear estimation problems with noises having pointwise and distributed shifts are derived.

Key Words: Estimation, filtering, smoothing, prediction, duality, control.

1. Introduction

Estimation theory is a widely used concept in engineering studies such as space engineering, electronics, geophysics, etc. The estimation problems consist in estimating an unobservable signal process $x$ at instant $t$ on the basis of observation data $z$ on time interval $[0, \tau]$. Depending on the relation of $t$ and $\tau$ the three kinds of the estimation problems are considered: (a) filtering, if $t = \tau$, (b) smoothing, if $t < \tau$ and (c) prediction, if $t > \tau$.

The underlying ideas of the linear estimation were defined by Kalman and Bucy [1,2]. In particular, in Kalman [1], the duality between the linear filtering and linear regulator problems is obtained. This result determines the general approach for synthesizing the optimal estimators through the optimal controls in the dual linear regulator problems. This approach is used in Bashirov and Mishne [3,4] for synthesizing the optimal filters in the linear filtering problems when the noises on the signal and the observations have pointwise and distributed shifts.

Application of the approach, based on duality, in studying the estimation problems requires the construction of the dual optimal control problem. The aim in this paper is the construction of the dual optimal control problems to the linear smoothing and prediction problems, with noises having pointwise and distributed shifts, and proving the duality theorem.
2. Notations

In this paper, the following notations are used:

\((\Omega, \mathcal{F}, P)\) : complete probability space

\(H, X, Z\) : real separable Hilbert spaces

\(\mathcal{L}(H, Z)\) : space of linear bounded operators on \(H\) to \(Z\)

\(< \cdot , \cdot >\) : inner product

\(\| \cdot \|\) : norm

\(B_2(a, b; \mathcal{L}(H, X))\) : class of strongly measurable \(\mathcal{L}(H, X)\)-valued functions \(F\) on \([a, b]\) with \(\int_a^b \| F_t \|^2 \, dt < \infty\)

\(B_\infty (a, b; \mathcal{L}(H, X))\) : class of strongly measurable \(\mathcal{L}(H, X)\)-valued functions \(F\) on \([a, b]\) with \(\text{ess sup} \| F_t \| < \infty\)

\(L_2(a, b; Z)\) : space of (equivalence classes of) Lebesgue measurable and square integrable functions on \([a, b]\) to \(Z\)

\(A^*\) : adjoint to operator \(A\)

\(\mathbb{E}\) : expectation

\(\text{cov}(x, y)\) : covariance operator of random variables \(x\) and \(y\)

\(\text{cov } x = \text{cov}(x, x)\)

\(\chi_{[a,b]}(s)\) : characteristic function of the set \([a, b]\)

3. Setting of Linear Estimation Problem for Shifted White Noises

Let \((x_t, z_t)\) be a partially observed linear stochastic system

\[
\dot{x}_t = Ax_t + \Phi w_t, \quad t > 0, \quad x_0 \text{ is given},
\]

\[
\dot{z}_t = C x_t + \Psi w_{t+\epsilon}, \quad t > 0, \quad z_0 = 0,
\]

where \(x_t\) and \(z_t\) are the signal and observation processes, respectively, at time \(t\), \(A\) is the infinitesimal generator of the strongly continuous semigroup \(\mathcal{U}_t\), \(t \geq 0\), \(C \in \mathcal{L}(X, Z), \Phi \in \mathcal{L}(H, X), \Psi \in \mathcal{L}(H, Z)\), \(w\) is \(H\)-valued white noise process with \(\mathbb{E} w_t = 0\), \(\text{cov}(w_t, w_s) = \overline{W} \delta(t-s)\) in which \(\delta\) is the Dirac’s delta function, \(x_0\) is a random variable with \(\mathbb{E} x_0 = 0\) and \(\text{cov} x_0 = P_0\). We suppose that \(x_0\) is independent of \(w\) and \(\epsilon > 0\). The system (1) - (2) can be written in the following integral form as well:

\[
x_t = \mathcal{U}_t x_0 + \int_0^t \mathcal{U}_{t-s} \Phi w_s ds, \quad t \geq 0,
\]
DUALITY BETWEEN CONTROL AND ESTIMATION

\[ z_t = \int_0^t C x_s ds + \int_0^t \Psi w_{s+\epsilon} ds, \quad t \geq 0. \]  \hspace{1cm} (4)

Let \( \tau > 0 \) and \( t > 0 \) be two time moments. The best linear estimation \( \hat{x}_t \) of \( x_t \) on the basis of observations \( z_s \) on \( 0 \leq s \leq \tau \) is a random variable in the form

\[ \hat{x}_t = \int_0^\tau K_s z_s ds; \quad K \in B_2(0, \tau; \mathcal{L}(Z, X)), \]  \hspace{1cm} (5)

that minimizes the error

\[ E \| x_t - \int_0^\tau G_s \hat{z}_s ds \|^2 \]  \hspace{1cm} (6)

over all \( G \in B_2(0, \tau; \mathcal{L}(Z, X)) \).

**Lemma 1**: Under the above conditions \( \hat{x}_t \), defined in (5), is a best linear estimation for the system (3) and (4), if and only if \( K \) satisfies the following Wiener-Hopf equation

\[ K_s V + \int_0^\tau K_r C \Lambda_{r,s} C^* dr + \int_0^{\max(0, s-\epsilon)} K_r N^* U_{s-r-\epsilon}^* C^* dr + \int_{\min(r, s+\epsilon)}^\tau K_r C U_{r-s-\epsilon} N dr = \Lambda_{t,s} C^* + \chi(\{0, t-\epsilon\}) (s) U_{t-s-\epsilon} N, \quad 0 \leq s \leq \tau, \]  \hspace{1cm} (7)

where

\[ \Lambda_{r,s} = U_r P_0 U_s^* + \int_0^{\min(r, s)} U_{r-\sigma} W U_{s-\sigma}^* d\sigma, \quad r \geq 0, \quad s \geq 0, \]  \hspace{1cm} (8)

and

\[ W = \Phi \bar{W} \Phi^*, \quad V = \Psi \bar{V} \Psi^*, \quad N = \Phi \bar{W} \Psi^*. \]  \hspace{1cm} (9)

**Proof.** By the orthogonal projection lemma (see Curtain [5]) (5) is the best linear estimation of \( x_t \) on the base \( z_s \), \( 0 \leq s \leq \tau \), if and only if

\[ \text{cov} \left( x_t - \hat{x}_t, \int_0^\tau G_s \hat{z}_s ds \right) = 0, \quad \text{for all } G \in B_2(0, \tau; \mathcal{L}(Z, X)). \]

Evaluating the above equality and using the arbitrariness of \( G \) one can obtain the equation (7) with (8) and (9) for \( K \) and vice versa.

Note that the equation (7) is not constructive in synthesizing of \( K \). Nevertheless, we will use (7) in proving of the duality theorem.
4. Dual Control Problem in the Case of Shifted White Noises

Consider the control problem

\[ \xi_s = - \begin{cases} \mathcal{U}_{s}^{*}\max(0,\tau-t)f, & s \geq \tau - t \\ 0, & s < \tau - t \end{cases} + \]

\[ \int_{\max(0,t-\tau)}^{\max(s,t-\tau)} \mathcal{U}_{s-r}^{*} C^* \eta_r dr, \quad 0 \leq s \leq \max(\tau, t), \]

and

\[ J(\eta) = \left\langle \xi_{\max(\tau,t)}, P_0 \xi_{\max(\tau,t)} \right\rangle + \int_{0}^{\max(\tau,t)} \left\langle \xi_s, W \xi_s \right\rangle ds + \]

\[ \int_{\max(0,t-\tau)}^{\max(\tau,t)} \left\langle \eta_s, V \eta_s \right\rangle ds + 2 \int_{\max(\epsilon,t-\tau)}^{\max(\tau,t)} \left\langle \eta_s, N^* \xi_{s-\epsilon} \right\rangle ds, \]

where \( \xi \) is a state process, \( J(\eta) \) is the functional to be minimized, \( \eta \) is a control from the set of admissible controls \( L_2(\max(0, t - \tau), \max(\tau, t); Z) \). \( f \) is an arbitrary fixed vector in \( X \), \( N, V, W \) are defined in (9), \( U, P_0 \) and \( \epsilon \) are as defined in Section 3.

Note that in the control problem (10) - (11), the cases \( t - \tau = 0 \), \( t - \tau < 0 \) and \( t - \tau > 0 \) are available and in fact (10) - (11) is the combination of these three cases.

**Lemma 2:** Under the above conditions \( \eta \) is an optimal control in the control problem (10) - (11) if and only if it satisfies the following equation

\[ V \eta_s + \int_{\max(0,t-\tau)}^{\max(\tau,t)} C \Lambda^*_{\max(\tau,t)-\tau, \max(\tau,t)-s} \eta_r dr + \]

\[ \int_{\max(0,t-\tau)}^{\max(s+\epsilon,t-\tau)} N^* \mathcal{U}_{s-r-\epsilon}^{*} C^* \eta_r dr + \int_{s+\epsilon}^{\max(s+\epsilon,\tau,t)} C \Lambda^*_{\tau,s+\epsilon} \mathcal{U}_{s-\epsilon}^{*} N \eta_r dr = \]

\[ C \Lambda^*_{\max(\tau,t),s} f + \chi_{(\max(\epsilon,\epsilon+\tau-t),\infty)}(s) N^* \mathcal{U}_{s-\epsilon}^{*} \min(0, t-\tau) f, \]

where \( \max(0, t - \tau) \leq s \leq \max(\tau, t) \) and \( \Lambda^*_{\tau,s} \) is defined in (8).

**Proof.** If \( \eta \) is the optimal control then \( J(\eta + \lambda \Delta \eta) - J(\eta) \geq 0 \) for all real numbers \( \lambda \) and admissible controls \( \Delta \eta \). Evaluating the above inequality and using the arbitrariness of \( \eta \) and \( \Delta \eta \) one can obtain the equation (12) for \( \eta \) and vice versa. Note that the proof
of Lemma 2 in general case has difficulties. To overcome this, it is convenient to prove it for \( t - \tau < 0 \) and \( t - \tau > 0 \) cases separately. The case \( t - \tau = 0 \) needs no separate consideration since it can be seen in any of the above cases.

**Theorem 1**: Under the above conditions (5) is the best linear estimation for the system (3) - (4) if and only if \( \eta_s = K_{\max(\tau,t)-s}^* f \), \( \max(0,t-\tau) \leq s \leq \max(\tau,t) \) is an optimal control in the control problem (10) - (11).

**Proof.** Let \( \eta \) be optimal in the control problem (10) - (11). Then, by Lemma 2, it satisfies the equation (12). Substituting \( \eta_s = K_{\max(\tau,t)-s}^* f \) in (12), using arbitrariness of \( f \) and taking adjoint in both sides of (12) one can obtain that \( K \) satisfies (7). So, by Lemma 1, the best linear estimation for the system (3) - (4) has the form (5). Conversely, if (5) is the best linear estimation for the system (3) - (4), then by Lemma 1 \( K \) in (5) satisfies (7). Taking adjoint in both sides of (7), one can obtain that \( \eta_s = K_{\max(\tau,t)-s}^* f \) satisfies (12) for all \( f \in X \). So, by Lemma 2 the control \( \eta_s = K_{\max(\tau,t)-s}^* f \) is optimal in the control problem (10) - (11).

Theorem 1 states the duality between the estimation problem for the system (3) - (4) and the control problem (10) - (11). By this theorem synthesizing of \( K \) in (5) is equivalent to finding optimal control as in Theorem 1.

### 5. Setting of Linear Estimation Problem for Shifted White and Wide-Band Noises

In this and next section, the results of the previous sections will be modified to the linear estimation problem (13) - (14) defined below.

Consider the partially obsered linear stochastic system

\[
x_t = U_t x_0 + \int_0^t \int_{\max(0,t-\epsilon)}^s U_{t-s} \Phi_{\theta-s} w_\theta d\theta ds, \quad t \geq 0, \tag{13}
\]

\[
z_t = \int_0^t C x_s ds + \int_0^t \Psi w_s ds, \quad t \geq 0, \tag{14}
\]

where the conditions of Section 3 hold and \( \Phi \in B_\infty(-\epsilon,0; L(H,X)) \). One can verify that the noise

\[
\varphi_t = \int_{\max(0,t-\epsilon)}^t \Phi_{\theta-t} w_\theta d\theta \tag{15}
\]

of the signal system (13) is obtained by distributed shift of the observation noise and satisfies

\[
\text{cov}(\varphi_t, \varphi_s) = \begin{cases} 0, & |t-s| \geq \epsilon \\ \neq 0, & |t-s| < \epsilon. \end{cases} \tag{16}
\]
So Φ is a wide-band noise and one can specify the system (13) - (14) as a system with shifted white and wide-band noises. Note that in the case of system (3) - (4) the pointwise shift of the white noises was used.

The following result can be proved similar to the proof of Lemma 1.

Lemma 3 : \( \hat{z}^*_t \), defined in (5), is a best linear estimation for the system (13) - (14) if and only if \( K \) satisfies the following Wiener-Hopf equation

\[
K_s V + \int_0^\tau K_r C C^* dr + \int_0^\tau \int_{\max(-\varepsilon, r-s)}^0 K_r N_\sigma \eta^{*}_s C^* d\sigma dr + \int_s^\tau \int_{\max(-\varepsilon, s-r)}^0 K_r C U_{r-s+\sigma} N_\sigma d\sigma dr = \Lambda_t, (17)
\]

where

\[
\Lambda_t = U_t P_0 U_t^{*} + \int_0^{\min(r,s)} \int_{\max(-\varepsilon, r-s)}^0 \int_{\max(-\varepsilon, \theta-s)}^0 U_{r-\theta+\sigma} W_{\sigma,\alpha} U_{\theta+\sigma} d\alpha d\sigma d\theta, \quad (18)
\]

\[
W_{\sigma,\alpha} = \Phi_\sigma \overline{W} \Phi_\alpha^*, \quad N_\theta = \Phi_\theta \overline{W} \Psi^*, \quad V = \Psi \overline{W} \Psi^*. \quad (19)
\]

6. Dual Control Problem in the Case of Shifted White and Wide-Band Noises

Consider the state process \( \xi \), defined in (10), and the functional, where \( \eta \) is a control from the set of admissible controls \( L_2(\max(0, t - \tau), \max(\tau, t); Z) \) and \( W, N, V \) are as defined in (19).

The following result can be proved similar to the proof of Lemma 2.

Lemma 4 : Under the above conditions \( \eta \) is an optimal control in control problem (10) and (20) if and only if it satisfies the following equation

\[
V \eta_s + \int_{\max(0, t-s)}^{\max(\tau, t)} \int_{\max(-\varepsilon, r-s)}^0 C \Lambda^*_{\max(\tau, t)-r, \max(\tau, t)-s} C^* \eta_r d\sigma dr + \int_s^{\max(0, t-s)} \int_{\max(-\varepsilon, r-s)}^0 N_\sigma \eta^{*}_s C^* \eta_r d\sigma dr + \]

18
\[
\int_s^{\max(\tau, t)} \int_0^{\max(-\epsilon, s-r)} \mathcal{U}_{r-s} N_{\sigma} \eta_r d\sigma dr = \int_{\max(0, t-s)}^{\max(\tau-t-s, \max(0, \tau-t))} \mathcal{U}_{s+\theta} N_{\sigma} \eta_s d\theta
\]

where \(\max(0, t-\tau) \leq s \leq \max(\tau, t)\) and \(\Lambda_{\tau,s}\) is defined in (18).

**Theorem 2:** Under the above conditions (5) is the best linear estimation for the system (13) - (14) if and only if \(\eta_s = K_{\max(\tau, t)-s}^{*} \mathcal{U}_{s+\theta} \), \(\max(0, t-\tau) \leq s \leq \max(\tau, t)\), is an optimal control in the control problem (10) and (20).

**References**


**ÖZET**

Bu çalışmada, kestirim problemlerinden olan düzleme ve öngörü problemlerini ele alınmış ve bu problemlerin noktasal ve dağılımsız kaydırma içeren gürtülüler için çifte denetim problemi türetilmiştir. Kismen gözlemlenebilen doğrusal türel sistem için dikey izdüşüm ön kuramı ve artırım yöntemi kullanılmış ve en iyi doğrusal kestirim bulunmuştur.