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RESIDUAL LIFETIME OF A SYSTEM WITH A COLD STANDBY UNIT

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Abstract: In this paper, we define and study two different residual life random variables corresponding to a single unit system equipped with a cold standby unit. We obtain the conditional survival functions when the lifetimes of active and standby units are dependent. Some properties of the associated mean residual life functions are also investigated. Graphical illustrations are presented to observe time dependent behaviors of associated mean residual life functions.

Key words: Mean residual life; Reliability; Standby redundancy; Stochastic ordering; Survival function.
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1. Introduction

Although the concept of residual life has been well studied in the literature under active redundancy ([1], [2],[5],[6],[8],[10],[3],[13],[14],[16]), it has not been considered for a system with a standby unit. In the case of cold standby redundancy, the standby redundant component neither degrade nor fail while in standby. Levitin et al. [11] have studied the optimal loading cold standby systems. Eryilmaz [4] analyzed the distribution and the expected value of warm standby components for general coherent systems. Eryilmaz [9] investigated a k-out-of-n system with a single warm standby and obtained an explicit expression for the system reliability. Some recent contributions on reliability analysis of systems under cold standby redundancy are in [7],[18],[17].

Consider a single unit system equipped with a cold standby unit. Let Y and X denote respectively the lifetimes of active and inactive (standby) units. It is clear that the lifetime of the entire system corresponds to the random variable $T = Y + X$. The usual residual life of the system is then defined by

$$\{Y + X - t \mid Y + X > t\}, \tag{1.1}$$

for $t > 0$. The conditional random variable defined by (1.1) implies only the survival of the system at time t, but no information is included about which unit survives at time t. Define the following conditional random variables:

$$\{Y + X - t \mid Y \leq t, X > t\}, \tag{1.2}$$

and

$$\{Y + X - t \mid Y > t\}. \tag{1.3}$$

The condition in (1.3) may be changed to $\{Y > t, X > t\}$ since in the case of cold standby redundancy $P\{Y > t\} > 0$ implies $P\{X > t\} = 1$ for $t > 0$. The conditional random variable defined by (1.2) represents the residual life of the system given that the active unit has failed before time t but the system works at time t with standby unit. Similarly, (1.3) represents the residual life of the

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system under the condition that the system works at time t with the active component. Obviously, the definitions (1.2) and (1.3) are more informative than (1.1).

In the present paper, we study the conditional random variables defined by (1.2) and (1.3) when the lifetimes of the units are dependent. In Section 2, we obtain survival and mean residual life functions corresponding to (1.2) and (1.3). Section 3 includes some graphical illustrations for the mean residual life functions.

2. Conditional survival functions and mean residual lifetimes

Let X and Y be dependent lifetime random variables with the joint cumulative distribution function (c.d.f.) $H(x,y) = P\{X \le x, Y \le y\}$, and marginal distributions $G(y) = P\{Y \le y\}$ and $F(x) = P\{X \le x\}$, for $x, y > 0$.

In the following, we derive the survival function of the conditional random variable (1.2).

THEOREM 1. *The conditional survival function of* $Y+X$ *given* $\{Y \le t, X > t\}$ *is*

$$\begin{aligned} &P\{Y+X>s \mid Y \le t, X>t\} \\ &= \frac{1}{G(t)-H(t,t)}\left[\int_0^{s-t} P\{X>s-y \mid Y=y\}\, dG(y) \right. \\ &\left. + \int_{s-t}^{t} P\{X>t \mid Y=y\}\, dG(y)\right], \end{aligned}$$

for $t \le s < 2t$, *and*

$$\begin{aligned} &P\{Y+X>s \mid Y \le t, X>t\} \\ &= \frac{1}{G(t)-H(t,t)}\left[\int_0^{t} P\{X>s-y \mid Y=y\}\, dG(y)\right], \end{aligned}$$

for $s \ge 2t$.

PROOF. By conditioning on Y,

$$\begin{aligned} &P\{Y+X>s, Y \le t, X>t\} \\ &= \int_{y \le t} P\{X>s-y, X>t \mid Y=y\}\, dG(y) \\ &= \int_{y \le t, s-y>t} P\{X>s-y \mid Y=y\}\, dG(y) + \int_{y \le t, s-y<t} P\{X>t \mid Y=y\}\, dG(y) \\ &= \int_0^{\min(t,s-t)} P\{X>s-y \mid Y=y\}\, dG(y) + \int_{s-t}^{t} P\{X>t \mid Y=y\}\, dG(y). \end{aligned}$$

Thus the required result is obtained considering the cases $s-t<t$ and $s-t \ge t$, and noting that

$$\begin{aligned} P\{Y \le t, X>t\} &= P\{Y \le t\} - P\{X \le t, Y \le t\} \\ &= G(t) - H(t,t). \blacksquare \end{aligned}$$

The expected value of random variable (1.2) represents the mean residual life of the system and it can be computed from Theorem 1 as

$$\begin{aligned} m(t) &= E(Y+X-t \mid Y \le t, X>t) \\ &= \int_0^{\infty} P\{Y+X>t+u \mid Y \le t, X>t\}\, du. \end{aligned}$$

COROLLARY 1. *If X and Y are independent, then*

$$P\{Y+X>s \mid Y\le t, X>t\} = \begin{cases} \frac{1}{G(t)\bar{F}(t)}\left[\int_0^{s-t}\bar{F}(s-y)dG(y)+\int_{s-t}^{t}\bar{F}(t)dG(y)\right], & \text{if } t\le s<2t \\ \frac{1}{G(t)\bar{F}(t)}\int_0^{t}\bar{F}(s-y)dG(y), & \text{if } s\ge 2t. \end{cases}$$

COROLLARY 2. *If X and Y are independent, then*

$$m(t)=t-\int_0^t \frac{G(x)}{G(t)}dx+\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}dx.$$

PROOF. If X and Y are independent, then

$$m(t)=E(Y\mid Y\le t)+E(X-t\mid X>t).$$

It is clear that

$$P\{Y>x\mid Y\le t\}=\frac{G(t)-G(x)}{G(t)},$$

for $x\le t$, and

$$E(Y\mid Y\le t)=\int_0^t \frac{G(t)-G(x)}{G(t)}dx=t-\int_0^t \frac{G(x)}{G(t)}dx.$$

On the other hand,

$$E(X-t\mid X>t)=\frac{1}{\bar{F}(t)}\int_0^\infty \bar{F}(t+x)dx.$$

Thus the proof is completed.■

The function defined by

$$\alpha_G(t)=E(t-Y\mid Y\le t)=\int_0^t \frac{G(x)}{G(t)}dx$$

is called the mean inactivity time (MIT). G_1 is said to be smaller than G_2 in the mean inactivity time order (denoted by $G_1\le_{MIT}G_2$) if $\alpha_{G_1}(t)\ge\alpha_{G_2}(t)$ for all $t>0$. See, e.g. [12] for the mean inactivity time order.

Let $\beta_F(t)=E(X-t\mid X>t)$ denote the MRL function corresponding to the distribution F. F_1 is said to be smaller than F_2 in the mean residual life time order (denoted by $F_1\le_{MRL}F_2$) if $\beta_{F_1}(t)\le\beta_{F_2}(t)$ for all $t>0$. The reader is referred to Shaked and Shanthikumar [15] for the details of various stochastic orderings.

Let $m_{G|F}(t)$ denote the MRL function of the system composed of an active unit with its distribution function G and a standby unit with its distribution F, respectively. If X and Y are independent, then from Corollary 2 we have

$$m_{G|F}(t)=t-\alpha_G(t)+\beta_F(t). \tag{2.1}$$

The proofs of the following results are immediate from (2.1).

PROPOSITION 1. *Suppose that X and Y are independent. Then,*
(a) if $G_1\le_{MIT}G_2$, then $m_{G_1|F}(t)\le m_{G_2|F}(t)$ for all $t>0$.

(b) if $F_1 \leq_{MRL} F_2$, then $m_{G|F_1}(t) \leq m_{G|F_2}(t)$ for all $t>0$.

PROPOSITION 2. *Suppose that X and Y are independent. If F is IMRL (increasing mean residual life) and G is DMIT (decreasing mean inactivity time), then $m_{G|F}(t)$ is nondecreasing in t.*

In the following, we derive the survival function of the conditional random variable defined by (1.3).

THEOREM 2. *The conditional survival function of $Y+X$ given $\{Y>t\}$ is*

$$\begin{aligned}&P\{Y+X>s \mid Y>t\}\\ &= \frac{1}{\bar{G}(t)}\left[\bar{G}(s)+\int_t^s P\{X>s-y \mid Y=y\}\, dG(y)\right],\end{aligned}$$

for $s>t$.

PROOF. For $s>t$, one can write

$$\begin{aligned}P\{Y+X>s, Y>t\} &= \int_{y>t} P\{X>s-y \mid Y=y\}\, dG(y)\\ &= \int_{y>t, s-y<0} dG(y) + \int_{y>t, s-y>0} P\{X>s-y \mid Y=y\}\, dG(y)\\ &= \bar{G}(s)+\int_t^s P\{X>s-y \mid Y=y\}\, dG(y).\end{aligned}$$

Thus the result follows.■

Define

$$u(t)=E(Y+X-t \mid Y>t).$$

Using Theorem 2,

$$\begin{aligned}u(t) &= \int_0^\infty P\{Y+X>t+z \mid Y>t\}\, dz\\ &= \int_0^\infty \frac{\bar{G}(t+z)}{\bar{G}(t)} dz + \frac{1}{\bar{G}(t)}\int_0^\infty \int_t^{t+z} P\{X>t+z-y \mid Y=y\}\, dG(y)dz\\ &= E(Y-t \mid Y>t)+\frac{1}{\bar{G}(t)}\int_0^\infty \int_t^{t+z} P\{X>t+z-y \mid Y=y\}\, dG(y)dz.\end{aligned}$$

COROLLARY 3. *If X and Y are independent, then*

$$u(t)=E(Y-t \mid Y>t)+E(X).$$

The proofs of the following results are immediate from Corollary 3.

PROPOSITION 3. *Suppose that X and Y are independent. Then,*
(a) if $G_1 \leq_{MRL} G_2$, then $u_{G_1|F}(t) \leq u_{G_2|F}(t)$ for all $t>0$.
(b) if $F_1 \leq_{ST} F_2$, then $u_{G|F_1}(t) \leq u_{G|F_2}(t)$ for all $t>0$.

PROPOSITION 4. *Suppose that X and Y are independent. If G is IMRL (DMRL), then $u_{G|F}(t)$ is nondecreasing (nonincreasing) in t.*

3. Graphical illustrations

For an illustration, let

$$H(x,y)=(1-e^{-\lambda x})(1-e^{-\lambda y})\left\{1+\theta e^{-\lambda x}e^{-\lambda y}\right\},$$

for $x,y\geq 0$ and $-1\leq\theta\leq 1$. That is, the joint distribution of X and Y is modeled by FGM (Farlie-Gumbel-Morgenstern) type bivariate exponential distribution. This model includes both positive dependence (for $\theta\geq 0$) and negative dependence (for $\theta\leq 0$). The case of independence is obtained when $\theta=0$. Then, the conditional probability density function of X given $Y=y$ is

$$h(x\mid y)=\lambda e^{-\lambda x}\left\{1+\theta(2e^{-\lambda x}-1)(2e^{-\lambda y}-1)\right\},$$

and hence

$$\begin{aligned}P\{X>t\mid Y=y\}&=\int_t^\infty h(x\mid y)dx\\&=e^{-\lambda t}\left\{1+\theta(1-e^{-\lambda t})(1-2e^{-\lambda y})\right\}.\end{aligned}$$

Under these assumptions, the functions $m(t)$ and $u(t)$ are found to be

$$m(t)=\frac{e^{-\lambda t}\left[(4+5\theta-6\theta e^{-\lambda t}+2\theta e^{-2\lambda t})+2\lambda t(1+\theta-2\theta e^{-\lambda t}+\theta e^{-2\lambda t})\right]-4-\theta}{2\lambda(e^{-\lambda t}+\theta e^{-\lambda t}-2\theta e^{-2\lambda t}+\theta e^{-3\lambda t}-1)},$$

$$u(t)=\frac{1}{2\lambda}\left(\theta-\theta e^{-\lambda t}+4\right),$$

In Figures 1-6, we plot the functions $m(t)=E(Y+X-t\mid Y\leq t,X>t)$ and $u(t)=E(Y+X-t\mid$

$Y>t)$ for various values of dependence parameter θ when $\lambda=1$. As expected, the function $m(t)$ is greater than $u(t)$ for all $t>0$. From the figures, we see that the more dependence between X and Y, the more apart from the independence case. One can also observe from Figures 1-3 that the statement of Proposition 2 is not generally true when X and Y are dependent.

For $\theta=0$, the random variables X and Y are independent and each have an exponential distribution. Therefore $E(Y-t\mid Y>t)$ is independent of t, and hence $u(t)$ is constant when $\theta=0$.

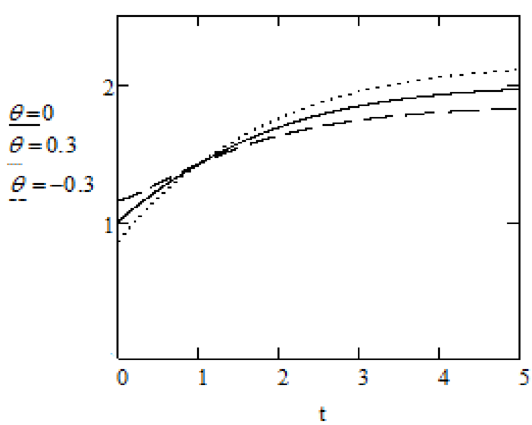

Figure 1. Plot of $m(t)$ for various values of θ.

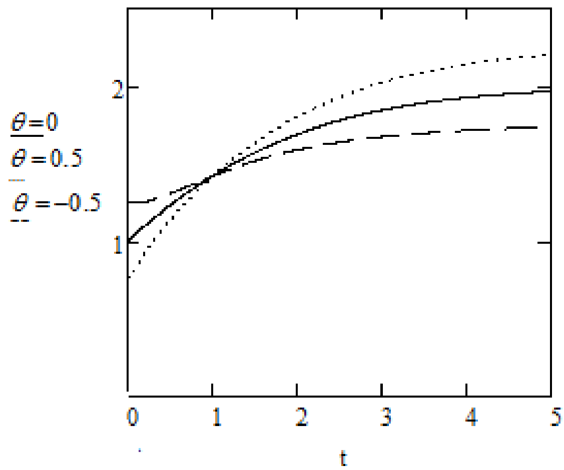

Figure 2. Plot of $m(t)$ for various values of θ.

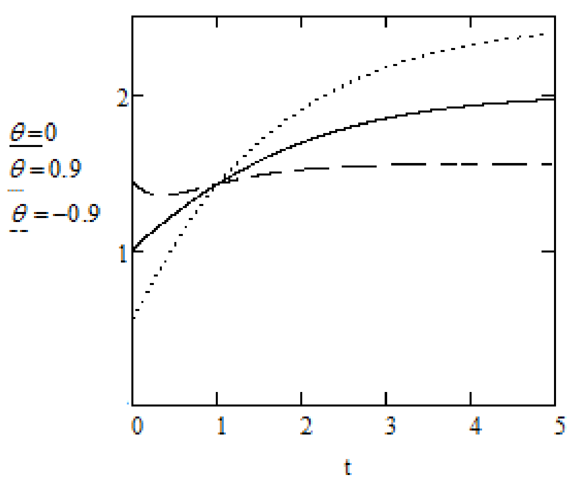

Figure 3. Plot of $m(t)$ for various values of θ.

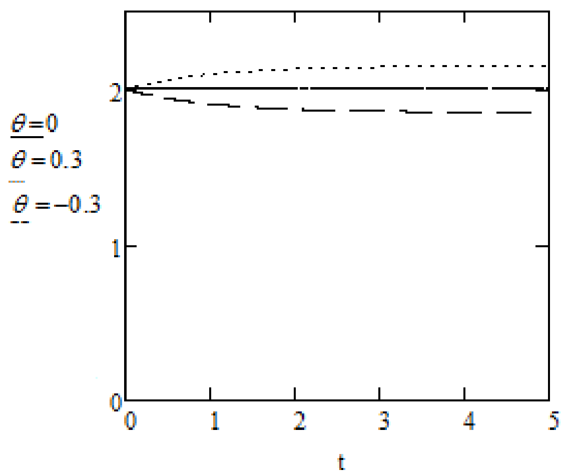

Figure 4. Plot of $u(t)$ for various values of θ.

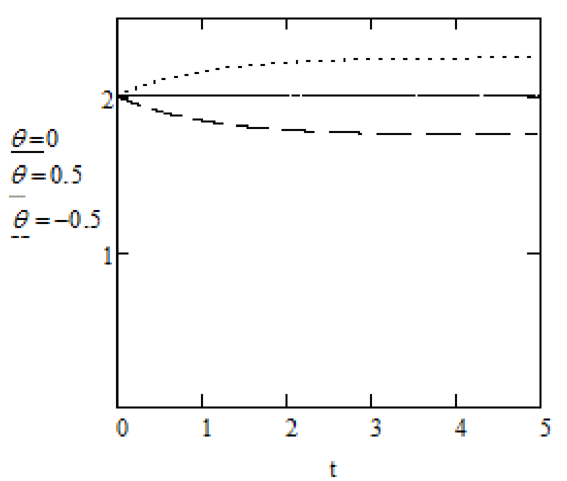

Figure 5. Plot of $u(t)$ for various values of θ.

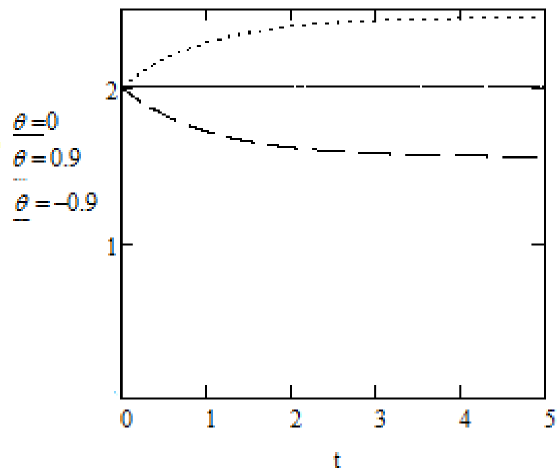

Figure 6. Plot of $u(t)$ for various values of θ.

4. Concluding Remarks

In this paper, we have introduced two different residual life functions for a single unit system equipped with a cold standby unit. We have presented the corresponding conditional survival functions when the lifetimes of active and standby units are dependent, and investigated some properties of MRL function under independence. The results of this paper can be generalized to a system equipped with k standby units.

If a system is equipped with k standby units with respective lifetimes $X_1, ..., X_k$, then a more general residual life can be defined as

$$\{Y + X_1 + ... + X_k - t \,|\, Y \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+1} > t\},$$

for $i = 0, 1, ..., k-1$. If the random variables $Y, X_1, ..., X_k$ are independent with $F_j(x) = P\{X_j \leq x\}, j = 1, ..., k$, then

$$m_k(t) = E(Y + X_1 + ... + X_k - t \,|\, Y \leq t, X_1 \leq t, ..., X_i \leq t, X_{i+1} > t),$$

$$= E(Y \,|\, Y \leq t) + \sum_{j=1}^{i} E(X_j \,|\, X_j \leq t) + \sum_{j=i+2}^{k} E(X_j) + E(X_{i+1} - t \,|\, X_{i+1} > t)$$

$$= (i+1)t - \alpha_G(t) - \sum_{j=1}^{i} \alpha_{F_j}(t) + \beta_{F_{i+1}}(t) + \sum_{j=i+2}^{k} E(X_j),$$

for $i = 0, 1, ..., k-1$, where $\alpha_{F_j}(t)$ and $\beta_{F_j}(t)$ denote respectively the MIT and MRL functions corresponding to the c.d.f. F_j.

As a possible future work, the residual lifetime of a system with a warm standby unit can also be studied.

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