RECURRENT RELATION FOR THE MOMENTS OF ORDER STATISTICS FROM A BETA-PARETO DISTRIBUTION

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Abstract: In this paper, a novel cumulative distribution function (c.d.f.) for beta-pareto (BP) distribution, through two distinct practical frames, is developed. However, the presented models are obviously more pragmatic than the ones being demonstrated in previous works, in the case of extending the further relations. Then, using the exhibited c.d.f.s, certain recurrence relations for the single and product moments of the order statistics of a random sample of size n arising from beta-Pareto distribution are derived.

Key words: Order statistics; Single and product moments; Recurrence relations; Beta-Pareto.

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1. Introduction

The class of generalized beta (GB) distribution was first introduced by [6] through its cumulative distribution function (c.d.f.). The c.d.f. of the class of GB distribution is defined by

$$F(x) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha > 0, \beta > 0,$$

(1.1)

The probability density function (p.d.f.), when X is continuous, for the GB distribution is given by

$$f(x) = \frac{1}{B(\alpha, \beta)} g(x) G(x)^{\alpha-1} (1 - G(x))^{\beta-1},$$

(1.2)

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Since the paper by [6] in which the normal distribution was defined and studied, many papers have appeared in this class. These include beta-frechet [11], beta-weibull [7], beta-pareto (BP) [1], \(\beta\)-birnbaum-saunders [4], and beta-cauchy [2]. The beta-Pareto (BP) distribution [1] was applied to fit flood data.

The order statistics have been extensively studied in the previous literature. For an excellent review, we refer to [2]; [3] and [5]. The use of recurrence relations for the moments of order statistics is quite well-known in statistical literature (see, for example, [2];[10]). For improved forms of these
results, it can be seen that [12] and [13], [3] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Hence, the aim of this paper is to consider order statistics of a random sample of size n drawn from BP distribution and derive some recurrence relations for the single and product moments of these order statistics.

Let $g(x)$ and $G(x)$ respectively be the p.d.f. and c.d.f. of pareto distribution, are given by

$$g(x) = \frac{k\theta^k}{x^{k+1}}, \quad k > 0, \theta > 0, x \geq \theta,$$

$$G(x) = 1 - \left(\frac{x}{\theta}\right)^{-k}, \quad k > 0, \theta > 0, x \geq \theta.$$

By replacing (1.3) and (1.4) in (1.2), the p.d.f. for the BP random variable is given by

$$f(x) = \frac{k}{\theta B(\alpha, \beta)} \left\{ 1 - \left(\frac{x}{\theta}\right)^{-k} \right\}^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta - 1}, \quad \alpha, \beta, \theta, k > 0, x \geq \theta.$$

This case is denoted by $X \sim BP(\alpha, \beta, \theta, k)$. The c.d.f. of BP random variable is denoted as $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics of a random sample (r.s.) of size n drawn from the BP distribution with p.d.f. (1.5). Then the p.d.f. for the $i-th$ order statistic, $X_{i:n}$, is given by

$$f_{i:n}(x) = \binom{n}{i} F(x)^{i-1} [1 - F(x)]^{n-i} f(x),$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

The $r-th$ moment of $X_{i:n}$ for $1 \leq i \leq n$ is as follows,

$$\mu_{i:n}^{(r)} = \binom{n}{i} \int_{\theta}^{\infty} x^r F(x)^{i-1} [1 - F(x)]^{n-i} f(x) dx.$$

The joint p.d.f. for the $i-th$ and $j-th$ order statistic, $(X_{i:n}, X_{j:n})$ is given by

$$f_{i,j:n}(x, y) = c_{i,j:n} F(x)^{i-1} [F(y) - F(x)]^{j-i-1} [1 - F(y)]^{n-j} f(x) f(y).$$

The $(r, s) - th$ product moment of $(X_{i:n}, X_{j:n})$ for $n \geq 2$ and $1 \leq i < j \leq n$, denoted by $\mu_{i,j:n}$, is given by

$$\mu_{i,j:n}^{(r,s)} = c_{i,j:n} \int_{\theta}^{\infty} \int_{x}^{\infty} x^r y^s F(x)^{i-1} [F(y) - F(x)]^{j-i-1} [1 - F(y)]^{n-j} f(x) f(y) dy dx.$$

where

$$c_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}.$$

In the following sections, we consider order statistics of a r.s. of size n arising from BP distribution with integer values for the shape parameters $\alpha$ and $\beta$. In section 2, we provide some extensions and properties of the c.d.f. for the BP distribution through two distinct frames. In section 3, some recurrence relations on the single moments of these order statistics are derived. Section 4 contains some discussions for the recurrence relations on the product moments of the order statistics.
Remark 1. In this article for any real number $\alpha$ and any natural $k$, we use the notation $\binom{\alpha}{k}$ for the generalized binomial coefficient

$$
\frac{\alpha(\alpha-1)...(\alpha-k-1)}{k!}.
$$

Note that by Newton’s generalization of binomial expansion, for any $x$ with $|x| < 1$, we have

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k. $$

2. Some extensions and properties

In this section we present some representations of the c.d.f. of BP distribution. The mathematical relation given in below will be useful in this, and next, section. If $\alpha$ is a positive real non-integer and $|z| < 1$, then ([8], p. 25)

$$(1 - z)^{\alpha-1} = \sum_{j=0}^{\infty} w_j z^j,$$

and if $\alpha$ is a positive real integer, then the upper of this summation stops at $\alpha - 1$, where

$$w_j = (-1)^j \binom{\alpha-1}{j} = \frac{(-1)^j \Gamma(\alpha)}{\Gamma(\alpha - j) \Gamma(j + 1)}.$$ 

Proposition 1. We can express (1.1) as a mixture of distribution function of BP distribution as follows: If $\alpha, \beta, \theta, k > 0$ and $j$ is positive integer we have,

$$F(x) = \sum_{j=0}^{\infty} C_1 x^{-k(j+\beta)} + C_2,$$  \hspace{1cm} (2.1)

where

$$C_1 = \frac{\theta^{k(j+\beta)}(-1)^{j+1} \binom{\alpha-1}{j}}{B(\alpha, \beta)(j + \beta)} \quad \& \quad C_2 = \sum_{j=0}^{\infty} \frac{(-1)^j \binom{\alpha-1}{j}}{B(\alpha, \beta)(j + \beta)}.$$  \hspace{1cm} (2.2)

Proof. The following relations are presented using (1.5) and the generalized binomial expansion by considering $\alpha$ is a positive real non-integer and $\left|\frac{x}{\theta}\right| < 1 ([8], p. 25)$.

$$F(x) = \int_{0}^{x} \frac{k}{B(\alpha, \beta)} \left\{ 1 - \left(\frac{x}{\theta}\right)^{-k(j+\beta)} \right\}^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta-1} dt$$

$$= \frac{k}{B(\alpha, \beta)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \theta^{k(j+\beta)} \int_{0}^{x} t^{-k(j+\beta)-1} dt$$

$$= \frac{k}{B(\alpha, \beta)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \theta^{k(j+\beta)} \left\{ \frac{(-1)^j \binom{\alpha-1}{j}}{k(j+\beta)} \left( x^{-k(j+\beta)} - \theta^{-k(j+\beta)} \right) \right\}$$

$$= \sum_{j=0}^{\infty} \frac{\theta^{k(j+\beta)}(-1)^{j+1} \binom{\alpha-1}{j}}{B(\alpha, \beta)(j + \beta)} x^{-k(j+\beta)} + \sum_{j=0}^{\infty} \frac{(-1)^j \binom{\alpha-1}{j}}{B(\alpha, \beta)(j + \beta)}.$$ 

The simplification of the preceding equation results in driven of (2.1).
Proposition 2. We can express (1.1) as a mixture of distribution function of \( BP \) distribution as follows:

\[
\begin{align*}
(I) \quad F(x) &= \sum_{t=0}^{\infty} b_t G(x)^t, \\
(II) \quad F(x)^n &= \sum_{t=0}^{\infty} d_{n,t} G(x)^t,
\end{align*}
\]

where

\[
b_t = \sum_{j=0}^{\infty} \sum_{l=t}^{\infty} p_j (-1)^{j+l} \binom{\alpha + j}{l} \binom{l}{t} \quad \text{and} \quad p_j = \frac{(-1)^j \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta - j) \Gamma(j + 1)(\alpha + j)},
\]

and \( d_{n,t} \) for \( t = 1, 2, \ldots \) are the coefficients and are easily determined from the recurrence equation

\[
d_{n,t} = (tb_0)^{-1} \sum_{m=1}^{t} [m(n + 1) - t] b_m d_{n,t-m},
\]

and \( d_{n,0} = b_0^2 \). Hence, \( d_{n,t} \) comes directly from \( d_{n,0}, \ldots, d_{n,t-1} \).

Proof. (I) Considering \([6]\) and by using \([8]\)

\[
F(x) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{1}{B(\alpha, \beta)} \sum_{j=0}^{\infty} \binom{\beta-1}{j} (-1)^j \int_0^{G(x)} t^{\alpha+j-1} dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{B(\alpha, \beta)(\alpha+j)} \binom{\beta-1}{j} G(x)^{\alpha+j} = \sum_{j=0}^{\infty} p_j G(x)^{\alpha+j},
\]

where

\[
p_j = \frac{(-1)^j \binom{\beta-1}{j}}{B(\alpha, \beta)(\alpha+j)} = \frac{(-1)^j \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta - j) \Gamma(j + 1)(\alpha + j)},
\]

also

\[
G(x)^{\alpha+j} = [1 - \{1 - G(x)\}]^{\alpha+j} = \sum_{l=0}^{\infty} (-1)^l \binom{\alpha+j}{l} (1 - G(x))^l = \sum_{l=0}^{\infty} (-1)^l \binom{\alpha+j}{l} \sum_{t=0}^{l} (-1)^t \binom{l}{t} G(t)^t,
\]

replacing \( \sum_{l=0}^{\infty} \sum_{t=0}^{l} \) by \( \sum_{t=0}^{\infty} \sum_{l=t}^{\infty} \) we obtain

\[
G(x)^{\alpha+j} = \sum_{t=0}^{\infty} \sum_{l=t}^{\infty} (-1)^{l+t} \binom{\alpha+j}{l} \binom{l}{t} G(x)^t,
\]

therefore

\[
F(x) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{t=l}^{\infty} p_j (-1)^{j+l} \binom{\alpha + j}{l} \binom{l}{t} G(x)^t = \sum_{t=0}^{\infty} b_t G(x)^t.
\]
and we obtain the result (I).

(II) Here and henceforth, we use an equation by [8], for a power series raised to a positive integer

\[
\left( \sum_{t=0}^{\infty} b_t x^t \right)^n = \sum_{t=0}^{\infty} d_{n,t} x^t,
\]

with this description, the proposition (II) is confirmed.

3. Recurrence relations for single moments

Throughout this section we assume that both \( \alpha \) and \( \beta \) are positive non-integers. Now we prove the following results which express \( \mu^{(r)}_{i:n}, 1 \leq i \leq n \), in terms of the single moments of BP order statistics.

**Theorem 1.** Let \( n \geq 2 \) and \( r \geq 1 \) be integers. If \( \alpha, \beta, \theta, k > 0 \), we can write

\[
\mu^{(r)}_{1:n} = \frac{n}{n-1} \left\{ (1 - C_2) \mu^{(r)}_{1:n-1} - \sum_{j=0}^{\infty} C_1 \mu^{(r-k(j+\beta))}_{1:n-1} \right\}, \tag{3.1}
\]

where \( C_1 \) and \( C_2 \) according to the (2.2).

**Proof.** For \( n \geq 2 \), we may write the following using (1.6) and (1.7),

\[
\mu^{(r)}_{1:n} = E(X^r_{1:n}) = n \int_0^\infty x^r (1 - F(x))^{n-1} f(x) dx
\]

\[
= n \left\{ \int_0^\infty x^r (1 - F(x))^{n-2} f(x) dx - \int_0^\infty x^r (1 - F(x))^{n-2} F(x) f(x) dx \right\}
\]

\[
= n \left\{ \left( \frac{1}{n-1} \right) \mu^{(r)}_{1:n-1} - \int_0^\infty x^r (1 - F(x))^{n-2} \left( \sum_{j=0}^{\infty} C_1 x^{-k(j+\beta)} + C_2 \right) f(x) dx \right\}
\]

\[
= \frac{n}{n-1} \left\{ (1 - C_2) \mu^{(r)}_{1:n-1} - \sum_{j=0}^{\infty} C_1 \mu^{(r-k(j+\beta))}_{1:n-1} \right\}.
\]

**Theorem 2.** Let \( n \) and \( i \) be integers such that \( n \geq 2 \) and \( 2 \leq i \leq n \). If \( \alpha, \beta, \theta, k > 0 \), then we have

\[
\mu^{(r)}_{i:n} = \frac{n}{i-1} \left\{ \sum_{j=0}^{\infty} C_1 \mu^{(r-k(j+\beta))}_{i-1:n-1} + C_2 \mu^{(r)}_{i-1:n-1} \right\}, \tag{3.2}
\]

where \( C_1 \) and \( C_2 \) according to the (2.2).

**Proof.** By using (1.7) and (2.1) we have,
\( \mu_{i:n}^{(r)} = E(X_{i:n}^r) = i\binom{n}{i} \int_\theta^\infty x^r F(x)^{i-1} (1 - F(x))^{n-i} f(x) dx \)

\[ = i\binom{n}{i} \int_\theta^\infty x^r F(x) F(x)^{i-2} (1 - F(x))^{n-i} f(x) dx \]

\[ = i\binom{n}{i} \int_\theta^\infty x^r \left( \sum_{j=0}^\infty C_1 x^{-k(j+\beta)} + C_2 \right) F(x)^{i-2} (1 - F(x))^{n-i} f(x) dx \]

\[ = \frac{n}{i-1} \left\{ \sum_{j=0}^\infty C_1 \mu_{i-1:n-1}^{(r-k(j+\beta))} + C_2 \mu_{i-1:n-1}^{(r)} \right\}. \]

**Remark 2.** For \( n \geq 2 \),

\[ \mu_{n:n}^{(r)} = \frac{n}{n-1} \left\{ \sum_{j=0}^\infty C_1 \mu_{n-1:n-1}^{(r-k(j+\beta))} + C_2 \mu_{n-1:n-1}^{(r)} \right\}. \quad (3.3) \]

The value of \( \mu_{i:n}^{(r)} \) is presented in Table 1, for \( n \) up to 10, \( r \) up to 4, \( \alpha = 0.5, \beta = 2, k = 2 \) and \( \theta = 3 \).
Table 1. The moments of \( BP \) order statistics for \( n \) up to 10, \( \alpha = 0.5, \beta = 2, k = 2 \) and \( \theta = 3 \).

\[
\begin{array}{cccccc}
\hline
n & i & r = 1 & r = 2 & r = 3 & r = 4 \\
\hline
1 & 1 & 3.5343 & 13.4990 & 63.1312 & 621.5891 \\
2 & 1 & 3.1686 & 10.1250 & 32.7345 & 107.6848 \\
 & 2 & 3.9000 & 16.8731 & 93.5280 & 1135.4935 \\
3 & 1 & 3.0560 & 9.5464 & 29.6162 & 621.5891 \\
 & 2 & 3.3377 & 11.2821 & 38.9712 & 138.6457 \\
 & 3 & 4.1832 & 19.6865 & 120.8064 & 1633.9174 \\
4 & 1 & 3.0529 & 9.3295 & 28.5409 & 87.4219 \\
 & 2 & 3.1853 & 10.1973 & 32.8419 & 106.5516 \\
 & 3 & 3.4820 & 12.3670 & 45.1005 & 170.7398 \\
 & 4 & 4.4169 & 22.1024 & 1135.4935 & 2121.6432 \\
5 & 1 & 3.0361 & 9.2222 & 28.0273 & 85.2257 \\
 & 2 & 3.1203 & 9.7583 & 30.5955 & 96.2064 \\
 & 3 & 3.2828 & 10.8558 & 36.2115 & 203.1868 \\
 & 4 & 3.7347 & 14.3137 & 120.8064 & 2601.2573 \\
 & 5 & 4.6174 & 24.2844 & 192.4051 & 3074.3623 \\
6 & 1 & 3.0263 & 9.1608 & 27.7380 & 84.0132 \\
 & 2 & 3.0852 & 9.4737 & 29.4737 & 91.2885 \\
 & 3 & 3.1906 & 10.2157 & 32.8390 & 106.0421 \\
 & 4 & 3.3750 & 11.4959 & 39.5841 & 138.6457 \\
 & 5 & 3.7347 & 14.3137 & 120.8064 & 257.320 \\
 & 6 & 4.7939 & 26.2785 & 192.4051 & 3074.3623 \\
7 & 1 & 3.0200 & 9.1220 & 27.5576 & 83.2660 \\
 & 2 & 3.0638 & 9.3933 & 28.8202 & 88.4962 \\
 & 3 & 3.1387 & 9.8706 & 31.1076 & 98.2693 \\
 & 4 & 3.2598 & 10.6759 & 35.1474 & 116.4058 \\
 & 5 & 3.4615 & 12.1109 & 42.9117 & 154.3646 \\
 & 6 & 3.8440 & 15.1948 & 62.2819 & 203.1868 \\
 & 7 & 4.9523 & 26.2844 & 192.4051 & 3074.3623 \\
8 & 1 & 3.0158 & 9.0959 & 27.4370 & 82.7703 \\
 & 2 & 3.0497 & 9.3046 & 28.4016 & 86.7357 \\
 & 3 & 3.1061 & 9.6591 & 30.0759 & 93.7779 \\
 & 4 & 3.1931 & 10.2300 & 32.8272 & 105.7551 \\
 & 5 & 3.3266 & 11.288 & 37.4676 & 127.0565 \\
 & 6 & 3.5424 & 12.7001 & 46.1782 & 170.7494 \\
 & 7 & 3.9445 & 16.0263 & 67.6498 & 300.7888 \\
 & 8 & 4.9523 & 28.1258 & 214.0923 & 4005.0792 \\
9 & 1 & 3.0128 & 9.0775 & 27.3522 & 82.4235 \\
 & 2 & 3.0399 & 9.2355 & 28.1156 & 85.5452 \\
 & 3 & 3.0840 & 9.5187 & 29.4026 & 90.9025 \\
 & 5 & 3.2467 & 10.5767 & 34.5380 & 113.5381 \\
 & 6 & 3.3905 & 11.5704 & 39.7752 & 137.8712 \\
 & 7 & 3.6184 & 13.2650 & 49.3797 & 187.1885 \\
 & 8 & 3.9445 & 16.0263 & 67.6498 & 300.7888 \\
 & 9 & 5.2285 & 31.4841 & 255.2804 & 4464.0583 \\
10 & 1 & 3.0106 & 9.0639 & 27.2902 & 82.1707 \\
 & 2 & 3.0327 & 9.1994 & 27.9107 & 84.6984 \\
 & 3 & 3.0684 & 9.4200 & 28.9353 & 90.9321 \\
 & 4 & 3.1206 & 9.7390 & 30.4932 & 95.5002 \\
 & 5 & 3.2444 & 10.2267 & 32.8163 & 105.5715 \\
 & 6 & 3.3990 & 10.9268 & 36.3497 & 121.5048 \\
 & 7 & 3.4514 & 11.9994 & 42.0589 & 148.7822 \\
 & 8 & 3.6089 & 13.8075 & 52.5172 & 203.6483 \\
 & 9 & 4.1246 & 17.5672 & 77.9580 & 365.6455 \\
 & 10 & 5.3512 & 33.0305 & 274.9829 & 4919.4375 \\
\hline
\end{array}
\]
4. Recurrence relations for the product moments

In this section we prove the following results which express $\mu_{r,s,n}, 1 \leq r < s \leq n$, in terms of the product moments of BP order statistics.

**Theorem 3.** Let $\alpha, \beta, \theta, k > 0$. Then for $n \geq 2$ we have,

$$\mu_{n-1,n;n} = n \left\{ \left( \mu_{1:1}^{(1)} - D_2 \right) \mu_{n-1:n-1}^{(1)} - \sum_{j=0}^{\infty} D_1 \mu_{n-1:n-1}^{(2-k(j+\beta))} \right\},$$  \hspace{1cm} (4.1)

where

$$D_1 = \frac{\theta^j (\beta+j-\frac{1}{k})}{B(\alpha,\beta)(\beta+j-\frac{1}{k})} \quad \text{and} \quad D_2 = \sum_{j=0}^{\infty} \frac{\theta^j (\alpha-1)}{B(\alpha,\beta)(\beta+j-\frac{1}{k})}.$$  \hspace{1cm} (4.2)

**Proof.** From (1.8) we may write for $n \geq 2$,

$$\mu_{n-1,n;n} = E(X_{n-1:n}X_{n:n}) = n(n-1) \int_{\theta}^{\infty} \int_{x}^{\infty} xyF(x)^{n-2}f(x)f(y)dydx$$

$$= n(n-1) \int_{\theta}^{\infty} xF(x)^{n-2}f(x) \left\{ \int_{\theta}^{\infty} yf(y)dy \right\} dx$$

$$= n(n-1) \int_{\theta}^{\infty} xF(x)^{n-2}f(x)I_xdx,$$  \hspace{1cm} (4.3)

$$I_x = \int_{x}^{\infty} yf(y)dy = \int_{x}^{\infty} yf(y)dy - \int_{\theta}^{x} yf(y)dy = \mu_{1:1}^{(1)} - P_x,$$  \hspace{1cm} (4.4)

by using (1.5) and (4.4), we get

$$P_x = \int_{\theta}^{x} yf(y)dy = \int_{\theta}^{x} \frac{k}{\theta B(\alpha,\beta)} \left\{ \left( \frac{y}{\theta} \right)^{-k} \right\}^{(\alpha-1)} \left( \frac{y}{\theta} \right)^{-\beta-1} dy$$

$$\quad \quad = \sum_{j=0}^{\infty} \frac{(\alpha-1)}{j} \left( -1 \right)^{j+1} \frac{k}{\theta B(\alpha,\beta)} \theta^{k(j+\beta)+1} \int_{\theta}^{x} y^{-k(j+\beta)}dy$$

$$\quad \quad = \sum_{j=0}^{\infty} \frac{(\alpha-1)}{j} \left( -1 \right)^{j+1} \frac{k}{\theta B(\alpha,\beta)} \theta^{k(j+\beta)} \int_{\theta}^{x} y^{-k(j+\beta)+1} dy$$

$$\quad \quad = \sum_{j=0}^{\infty} \frac{(\alpha-1)}{j} \left( -1 \right)^{j+1} \frac{k}{\theta B(\alpha,\beta)} x^{-k(j+\beta)+1} + \sum_{j=0}^{\infty} \frac{(\alpha-1)}{j} \left( -1 \right)^{j+1} \frac{k}{\theta B(\alpha,\beta)} x^{-k(j+\beta)+1}$$

$$\quad \quad = \sum_{j=0}^{\infty} D_1 x^{-k(j+\beta)+1} + D_2,$$  \hspace{1cm} (4.5)

where $D_1$ and $D_2$ according to the (4.2). If we insert $P_x$ in the (4.4) then we can write

$$I_x = \mu_{1:1}^{(1)} - \sum_{j=0}^{\infty} D_1 x^{-k(j+\beta)+1} + D_2,$$

by using $I_x$ and (4.3), the theorem 3 proved.

**Theorem 4.** Theorem 4: Let $\alpha, \beta, \theta, k > 0$. Then for $n \geq 2$ and $1 \leq i < j \leq n$ we have,

$$\mu_{i,j;n}^{(r,s)} = \sum_{m=0}^{j-i-1} (-1)^m \binom{j-i-1}{m} c_{i,j;n} \frac{1}{c} \left\{ \binom{n-j}{m} \mu_{j-c,n-c}^{(r)} - \sum_{w=0}^{\infty} A_1 \mu_{j-c,n-c}^{(r)} - \sum_{w=0}^{\infty} A_2 \mu_{j-c,n-c}^{(r)} \right\},$$  \hspace{1cm} (4.6)
where \( c = i + m \) and

\[
A_1 = \sum_{w=0}^{\infty} \sum_{l=0}^{\infty} \sum_{t=0}^{\infty} d_{j-c+l-1,t} \frac{1}{B(\alpha, \beta)} (-1)^{l+w} \binom{t + \alpha - 1}{w} \binom{n - j}{l} \frac{\theta^s}{w + \beta - \frac{1}{k}}, \tag{4.7}
\]

\[
A_2 = \sum_{l=0}^{\infty} \sum_{t=0}^{\infty} d_{j-c+l-1,t} \frac{1}{B(\alpha, \beta)} (-1)^{l+w+1} \binom{t + \alpha - 1}{w} \binom{n - j}{l} \frac{g^{k(w+\beta)+s}}{w + \beta - \frac{1}{k}}. \tag{4.8}
\]

**Proof.** We know that

\[
[F(y) - F(x)]^{j-i-1} = \sum_{m=0}^{j-i-1} (-1)^{j-i-1} \binom{j-i-1}{m} F(x)^m F(y)^{j-i-1-m},
\]

then we have

\[
\mu_{i,j,n}^{(r,s)} = \sum_{m=0}^{j-i-1} (-1)^{j-i-1} \binom{j-i-1}{m} c_{i,j,n} \int_{\theta}^{x} \int_{y}^{\infty} x^{r} y^{s} F(x)^{i+m-1} F(y)^{j-i-1-m} [1 - F(y)]^{n-j} f(x) f(y) dy dx
\]

\[
= \sum_{m=0}^{j-i-1} (-1)^{j-i-1} \binom{j-i-1}{m} c_{i,j,n} I_x, \tag{4.9}
\]

where

\[
I_x = \int_{\theta}^{x} \int_{\theta}^{\infty} x^{r} y^{s} F(y)^{j-c-1} [1 - F(y)]^{n-j} f(y) dy
\]

\[
= \int_{\theta}^{x} \int_{\theta}^{\infty} y^{s} F(y)^{j-c-1} [1 - F(y)]^{n-j} f(y) dy - \int_{\theta}^{x} y^{s} F(y)^{j-c-1} [1 - F(y)]^{n-j} f(y) dy
\]

\[
= \frac{1}{(j-c)(n-c)} \mu_{j-c,n-c}^{(s)} - \int_{\theta}^{x} y^{s} F(y)^{j-c-1} [1 - F(y)]^{n-j} f(y) dy
\]

\[
= \frac{1}{(j-c)(n-c)} \mu_{j-c,n-c}^{(s)} - P_x, \tag{4.10}
\]

from (1.5) and (4.10) we can write,

\[
P_x = \int_{\theta}^{x} y^{s} F(y)^{j-c-1} [1 - F(y)]^{n-j} f(y) dy
\]

\[
= \int_{\theta}^{x} y^{s} F(y)^{j-c-1} \sum_{l=0}^{\infty} (-1)^{l} \binom{n-j}{l} F(y)^{l} f(y) dy
\]

\[
= \sum_{l=0}^{\infty} (-1)^{l} \binom{n-j}{l} \int_{\theta}^{x} y^{s} F(y)^{j-c+l-1} f(y) dy,
\]

by using (I) and (II) from proposition (II) we have

\[
F(y)^{j-c+l-1} = \left( \sum_{t=0}^{\infty} b_t G(y)^t \right)^{j-c+l-1} = \sum_{t=0}^{\infty} d_{j-c+l-1,t} G(y)^t = \sum_{t=0}^{\infty} d_{j-c+l-1,t} \left( 1 - \left( \frac{y}{\theta} \right) \right)^{-k},
\]
\[ P_z = \sum_{l=0}^{\infty} (-1)^l \binom{n-j}{l} \int_y y^s F(y)^{l+c+1} f(y) dy \]

\[ = \sum_{l=0}^{\infty} \sum_{t=0}^{\infty} d_{j-c+1-l,t} (-1)^t \binom{n-j}{l} \int_y y^s \left( 1 - \left( \frac{y}{\theta} \right)^{-k} \right)^t f(y) dy \]

\[ = \sum_{w=0}^{\infty} \sum_{l=0}^{\infty} \sum_{t=0}^{\infty} d_{j-c+1-l,t} (-1)^{t+w} \binom{n-j}{l} \left( t+\alpha-1 \right) \left( \frac{\theta^s}{\theta B(\alpha, \beta)} \right) \frac{k^{k(w+\beta)+1}}{\theta B(\alpha, \beta)} \int_y y^{-k(w+\beta)+s-1} dy \]

\[ = A_1 + \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s}, \]

where

\[ A_1 = \sum_{w=0}^{\infty} \sum_{l=0}^{\infty} \sum_{t=0}^{\infty} d_{j-c+1-l,t} (-1)^{t+w} \binom{n-j}{l} \left( t+\alpha-1 \right) \left( \frac{\theta^s}{\theta B(\alpha, \beta)} \right) \frac{k^{k(w+\beta)+1}}{\theta B(\alpha, \beta)} \]

\[ A_2 = \sum_{l=0}^{\infty} \sum_{t=0}^{\infty} d_{j-c+1-l,t} (-1)^{t+w} \binom{n-j}{l} \left( t+\alpha-1 \right) \left( \frac{\theta^{k(w+\beta)+s}}{w+\beta-\frac{1}{\hat{f}}} \right). \]

Substituting \( P_x \) in (4.10) we may write

\[ I_x = \frac{1}{(j-c)\binom{n-c}{j-c}} \mu_{j-c:n-c} - A_1 - \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s}, \]

Substituting \( P_x \) in (4.9) we may write

\[ \mu_{r,s}^{(r,s)} = \sum_{m=0}^{j-1} \binom{j-i-1}{m} c_{i,j:n} \frac{1}{x} \int_{\theta} x^r F(x) c-1 f(x) \left( \frac{(n-j)!}{(n-c)!} \mu_{j-c:n-c} - A_1 \right) \frac{\mu_{c, r}}{\mu_{c, c} - A_1} \frac{1}{x^{k(w+\beta)+s}} \right) dx \]

\[ = \sum_{m=0}^{j-1} \binom{j-i-1}{m} c_{i,j:n} \frac{1}{c} \left( \frac{(n-j)!}{(n-c)!} \mu_{j-c:n-c} - A_1 \right) \frac{\mu_{c, r}}{\mu_{c, c} - A_1} \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s+r} \right) \]

\[ = \sum_{m=0}^{j-1} \binom{j-i-1}{m} c_{i,j:n} \frac{1}{c} \left( \frac{(n-j)!}{(n-c)!} \mu_{j-c:n-c} - A_1 \right) \mu_{c, r} \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s+r} \right). \]
Table 2. The product moments of BP order statistics for $n$ up to 4, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$.

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The value of $\mu_{i,j;n}^{(r,s)}$ are presented in Table 2 and 3, for $n$ up to 5, $r$ and $s$ up to 4, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$. 
Table 3. The product moments of BP order statistics for \(n=5, \alpha = 0.5, \beta = 2, k = 2 \) and \( \theta = 3 \).

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5. Conclusion
We studied the certain recurrence relations for the single and product moments of the order statistics of a random sample of size \( n \) arising from beta-Pareto distribution are derived. It would be interest to a numerical application to illustrate the usefulness of the result in future work. The addition of this section was the kind suggestion of an anonymous referee of the journal.
References